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ON A FUNCTIONAL EQUATION GENERALIZING THE CLASS OF SEMISTABLE DISTRIBUTIONS

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Abstract. With $\varphi(p)$, $p \geq 0$ the Laplace-Stieltjes transform of some infinitely divisible probability distribution, we consider the solutions to the functional equation $\varphi(p) = e^{-p\beta} \prod_{i=1}^{m} \varphi^{\gamma_i}(c_i p)$ for some $m \geq 1$, $c_i > 0$, $\gamma_i > 0$, $i = 1, \dots, m$, $\beta \in \mathbb{R}$. We supply its complete solutions in terms of semistable distributions (the ones obtained when $m = 1$). We then show how to obtain these solutions as limit laws ($r \uparrow \infty$) of normalized Poisson sums of iid samples when the Poisson intensity $\lambda(r)$ grows geometrically with r.

Key words and phrases: Stable and semistable laws, functional equation, limit laws, selfsimilarity, generalized semistability.

1. Introduction

The origin of the problem is motivated by computing the class of non-degenerate infinitely divisible (ID) random variables bounded below whose Laplace-Stieltjes transform (LST) satisfies the functional equation

$$\varphi(p) = \prod_{i=1}^{m} \varphi^{\gamma_i}(c_i p) \tag{1.1}$$

for some $m \geq 1$, $c_i > 0$, $\gamma_i > 0$, $i = 1, \dots, m$. These may be considered as an extended class of one-sided semistable distributions which were introduced by Lévy (1937) (i.e. as those satisfying $\varphi(p) = \varphi^{\gamma}(cp)$ which is (1.1) with $m = 1$). In the literature, some related contributions to this field can be found in Ramachandran and Rao (1968), Shimizu (1978), for example. More specifically, in Shimizu (1978), a functional equation of this type was considered for characteristic functions and with the condition $\max |c_i| < 1$; here, neither positivity of the random variable involved nor of the coefficients $c_i > 0$, $i = 1, \dots, m$ were needed and $m < \infty$ was not even assumed; under these hypotheses, the infinite divisible character of the solutions was obtained directly from the functional equation. This constituted an ultimate generalization of a special case of known results.

One-sided semistable laws (or random variables) are identified with the ones of ID laws whose LST satisfies a functional equation of the form

$$\varphi(p) = e^{-p\beta} \varphi^{\gamma}(cp) \tag{1.2}$$

for some $c > 0$, $\beta \in \mathbb{R}$, $\gamma > 1$. This functional equation was first introduced by Lévy (1937) when φ is a characteristic function, leading to all semistable laws (see Lukacs (1983), p. 45 for a survey on this point and Sato (1999) for a recent monograph). In the sequel, we shall limit ourselves tacitly to one-sided semistable laws.

A class of distributions which contains the ones of stable (and semistable) distributions is then introduced; we call them generalized semistable (GSS) distributions. They constitute a sub-class of infinitely divisible distributions which are defined as the fixed point of the transformation on their LST displayed in (1.1). The aim of the manuscript is thus to give a complete solution to this equation in connection with semistable laws. Another objective is to discuss the very particular statistical status and properties which such distributions seem to entail. On this basis, these new models can be seen, in a natural way, as limit laws under a random geometric growth condition. Convergence Theorems established can be compared with the ones of Grinevich and Khokhlov (1993).

This work is then organized as follows.

In Section 2, the main features of semistable distributions are recalled (see Grinevich and Khokhlov (1993), Huillet *et al.* (2001), Kruglov (1972), Lévy (1937), Lukacs (1983), Pillai (1971), Sato (1999), Shimizu (1970) and the bibliography therein). These constitute a first-step extension to the one-sided Lévy-stable distribution in that their scale parameters are no longer constant but rather allowed to vary in a log-periodic fashion. The occurrence of semistable distributions as limit laws will briefly be recalled.

In Section 3, we shall exhibit the strong connections that exist between semistable and GSS distributions as solutions to (1.1). Central to the solution of this functional equation is the structure function

$$\tau(q) = \sum_{i=1}^{m} \gamma_i c_i^q, \qquad q \in \mathbb{R}$$

and the number $|\mathcal{S}_1|$, with $\mathcal{S}_1 = \{\alpha \in (0,1) : \tau(\alpha) = 1\}$. It will be shown that the values of α such that $\tau(\alpha) = 1$ are the characteristic exponents of one-sided α-semistable distributions appearing in the solutions of (1.1). So, the values of $\alpha \geq 1$ for which $\tau(\alpha) = 1$ have to be ruled out, as is done in $\mathcal{S}_1$.

More precisely, if $|\mathcal{S}_1| = 1$ and if $c_i = c^{r_i}$, $c \in (0,1)$, $r_i \in \mathbb{Z}$, one recovers the semistable class extending and including the one-sided stable distributions.

If $|\mathcal{S}_1| = 2$, one gets the full generalized semistable distributions. It is shown that the LST of such distributions is representable as the product of two semistable LSTs. Precise statements are summarized in Theorem 3.1 which follows from two preliminary results explained in Lemmas 3.1 and 3.2.

In Subsection 3.2, we shall show that it also makes sense to consider a random variable concentrated on $[x, \infty)$ for some $x > -\infty$ whose LST satisfies the functional equation

$$\varphi(p) = e^{-p\beta} \prod_{i=1}^{m} \varphi^{\gamma_i}(c_i p)$$

for some $m \geq 1$, $c_i > 0$, $\gamma_i > 0$, $i = 1, \ldots, m$, $\beta \in \mathbb{R}$. These can be obtained by shifting the solutions of the functional equation (1.1) by $x \in \mathbb{R}$, with $x = \beta/(1 - \sum_{i=1}^{m} \gamma_i c_i)$.

Some connections of GSS distributions with the notion of semi-selfsimilarity of induced processes are briefly emphasized in Subsection 3.3.

In Section 4, we finally show that GSS distributions may be seen as limit laws for some renormalized sum of an iid sample, when the number of terms in the sum grows geometrically with some parameter r. We proceed progressively in two steps. A primary deterministic illustration of geometrical growth is when the sample size $n(r) \sim K\rho^r$ for some constants $K > 0$, $\rho > 1$, when this number is generated by a deterministic version of the Crump and Mode (1968*a*, 1968*b*) branching process; the limiting laws in this case are identified with a particular generalized semistable description of semistable distributions, in accordance with known results from the literature. Main result on this aspect is Proposition 4.1. The full geometrical growth condition required to grasp all GSS distributions as limit laws must be random: the size $N(r)$ of the sample should be Poisson distributed whose intensity $\lambda(r) := \boldsymbol{E}[N(r)]$ has geometric growth. The main result is displayed in Proposition 4.2.

2. Semistable laws

We first briefly recall some results on semistable distributions.

2.1 *Strictly semistable law*

Let $\rho > 1$ and $c \in (0,1)$. First, consider the class of ID random variable X with support $[0,\infty)$ whose LST satisfies the simpler functional equation

$$\varphi_X(p) = \varphi_X^{\rho}(cp). \tag{2.1}$$

These variables will be identified with the so-called strictly semistable variables. According to Lévy-Khintchine formula (see Feller (1971), p. 450, Bertoin (1996)) there exists a measure π on $]0,+\infty[$ with $\int_0^{+\infty}(1\wedge x)\pi(dx) < \infty$ such that the Laplace-Stieltjes transform (LST) of X satisfy

$$\varphi_X(p) := \boldsymbol{E}e^{-pX} = \exp\left\{-\int_0^{+\infty}(1-e^{-px})\pi(dx)\right\}. \tag{2.2}$$

π is called the Lévy measure and letting $\pi(x) := -\pi(]x,+\infty[)$ we obtain the Lévy spectral function.

The solutions of (2.1) are well-known to be the class of ID random variables with spectral function

$$\pi(x) = -x^{-\alpha}s(\log x), \qquad x \in (0,+\infty), \tag{2.3}$$

where

(i) the constant $\alpha := -\log_c \rho$ necessarily belongs to the interval $(0,1)$,

(ii) $s(x)$ is a non-negative function such that $s(x) := e^{\alpha\nu(x)}$, for some right-continuous bounded periodic function ν with period $-\log c$, satisfying the additional condition: $x - \nu(x)$ is non-decreasing.

Turning back to the solution of (2.1), from (2.2) and (2.3), we obtain (see Huillet *et al.* (2001))

$$\varphi_X(p) = \exp\{-p^{\alpha}\tilde{s}(\log p)\} \tag{2.4}$$

where $\tilde{s}$ is a non-negative periodic function with period $-\log c$, the Fourier series expansion of which being easily obtained from the one of the function s which appears in the Lévy spectral function (2.3).

Remark 2.1. If in (2.3) the scale function $s(x) = s > 0$, a constant, we recover the Lévy stable laws. By the expression (2.4) the function $\tilde{s}$ is also constant, with $\tilde{s} := s\Gamma(1-\alpha)$ (see Uchaikin and Zolotarev (1999) for a recent overview of these subjects).

2.2 *Shifted semistable law*

Let $x \in \mathbb{R}$; consider the shifted variable $\widetilde{X} := X + x$. The shifted variable now satisfies the functional equation of the type (1.2), with $\beta = x(1 - \rho c)$. Conversely, if $\widetilde{X}$ is solution to (1.2), then $\widetilde{X} = X + x$ with $x = \beta/(1 - \rho c)$ and X is solution to (2.1). Clearly

$$\varphi_{\tilde{X}}(p) = \exp\{-[px + p^{\alpha}\tilde{s}(\log p)]\}, \qquad p \geq 0. \tag{2.5}$$

Random variables whose LST are given by (2.5) with $\tilde{s}(\cdot) = \tilde{s} > 0$, a constant, are known as shifted one-sided Lévy-stable variables.

2.3 *Semistable distributions as limit laws in Statistics*

Stable laws are well-known to be limit laws for sums of centered and normalized of n iid random variables (see Uchaikin and Zolotarev (1999) for more details and references). Allowing sample size to grow geometrically, semistable laws also appear as limit laws. Let X be a semistable variable whose LST is solution to (1.2). It appears as possible nondegenerate limit law of

$$\sum_{m=1}^{\rho_n} \frac{\mathcal{X}_m - x_n}{\sigma_n},$$

where $\mathcal{X}_m \stackrel{d}{=} \mathcal{X}$, $m \geq 1$ is an iid sequence, $x_n \in \mathbb{R}$, $\sigma_n > 0$ and $\rho_n > 0$ (see Pillai (1971)). The integer-valued sequence ρ_n is assumed to satisfy the additional geometrical growth properties: $\lim_{n\uparrow+\infty} \rho_n = +\infty$ and $\lim_{n\uparrow\infty} \rho_{n+1}/\rho_n = \rho \geq 1$.

The variable $\mathcal{X}$ is said to belong to the domain of partial attraction (DPA) of X. See Grinevich and Khokhlov (1993), Kruglov (1972), Shimizu (1970) for DPA characterization of general semistable laws.

3. The generalized semistable distributions

We now come to the related largest class of the GSS distributions.

3.1 *Generalized semistable laws*

Consider the class of ID random variables X with support $[a, \infty)$, $a > -\infty$, whose LST satisfy the functional equation (1.1)

$$\varphi_X(p) = \prod_{i=1}^{m} \varphi_X^{\gamma_i}(c_i p). \tag{3.1}$$

The following result yields a formal solution of equation (3.1).

LEMMA 3.1. *Suppose the LST of X solves (3.1). If $\sum_{i=1}^{m} \gamma_i c_i \neq 1$, necessarily X has support $[0, \infty)$. In any case, the Lévy spectral function of X has the formal representation*

$$\pi(x) = -\sum_{\alpha_l \in \mathcal{S}} x^{-\alpha_l} s_l(\log x) \tag{3.2}$$

where $\mathcal{S}$ is the set of solutions α to $\sum_{i=1}^{m} \gamma_i c_i^{\alpha} = 1$ and $s_l(\cdot)$ are non-negative periodic functions with periods $\log c_i$, $i = 1, \ldots, m$.

PROOF. By Lévy-Khintchine formula (see Feller (1971), p. 450, Bertoin (1996)), the Laplace exponent function of X reads $-\log \varphi_X(p) := ap + L(p)$, where $L(p) := \int_0^{+\infty} (1 - e^{-px}) \pi(dx)$ for some Lévy measure integrating $1 \wedge x$. By functional equation (3.1), we get

$$ap\left(1 - \sum_{i=1}^{m} \gamma_i c_i\right) + L(p) = \sum_{i=1}^{m} \gamma_i L(c_i p).$$

As $\lim_{p \uparrow \infty} L(p)/p = 0$, we get $a = 0$ if $\sum_{i=1}^{m} \gamma_i c_i \neq 1$.

Using an integration by parts,

$$L(p) = \int_0^{+\infty} (1 - e^{-px}) \pi(dx) = -p \int_0^{+\infty} \pi(x) e^{-px} dx,$$

we get in any case the scaling property for π

$$\pi(x) = \sum_{i=1}^{m} \gamma_i \pi(x/c_i).$$

Introducing the positive function $H(x) := -\pi(e^x)$, this functional equation takes the simpler convolution form

$$H(x) = \sum_{i=1}^{m} \gamma_i H(x + x_i), \qquad \forall x \in \mathbb{R}$$

with $x_i := -\log c_i$. Now, we are in the position to apply the Lau-Rao-Shanbhag theorem (see p. 38 Corollary 2.3.2 of Theorem 2.3.1 p. 36 of Rao and Shanbhag (1994)). Indeed, introduce the structure function $\tau(q) := \sum_{i=1}^{m} \gamma_i c_i^q$, $q \in \mathbb{R}$.

Under our hypothesis, it is positive and convex. As a result, the equation: $\tau(\alpha) = 1$ admits none, one or two solutions in $\mathbb{R}$. Denote by $\mathcal{S}$ the set of these solutions. Then, the function H takes the form

$$H(x) = \sum_{\alpha_l \in \mathcal{S}} e^{-\alpha_l x} s_l(x)$$

with the convention that the sum over the empty set is null. Here, the functions s_l are positive and periodic, that is satisfying $s_l(x) = s_l(x + x_i)$, for all $i = 1, \ldots, m$.

In terms of the Lévy spectral function π itself, we get, formally

$$\pi(x) = -\sum_{\alpha_l \in \mathcal{S}} x^{-\alpha_l} s_l(\log x), \qquad x \in (0, +\infty).$$

This completes the proof. □

As π must be the Lévy spectral function of some ID random variable X, additional conditions have to be imposed. We shall first need the following definition.

DEFINITION 3.1. For a given $\alpha \in \mathbb{R}$ and a positive function $s(\cdot)$, the pair $(\alpha, s(\cdot))$ will be called an admissible pair, if we have

(i) $\alpha \in (0,1)$,

(ii) function $s(\cdot)$ is representable as $s(x) := e^{\alpha\nu(x)}$, for some right-continuous bounded periodic function ν with periods $\boldsymbol{x} := -\log \boldsymbol{c}$, $\boldsymbol{c} := (c_1, \dots, c_m)$ such that $x - \nu(x)$ is non-decreasing function.

LEMMA 3.2. *Let X be a non-degenerate ID random variable whose Lévy spectral function is given by equation* (3.2) *then necessarily the pairs $(\alpha_l, s_l(\cdot))$, $l = 1,2$ are admissible. In other words, we have*

$$\pi(x) = -\sum_{\alpha_l \in \mathcal{S}_1} x^{-\alpha_l} s_l(\log x)$$

where $\mathcal{S}_1$ is the set of solutions α of $\tau(\alpha) = 1$ with values in $(0,1)$ and where for each $l = 1,2$ the pair $(\alpha_l, s_l(\cdot))$ is admissible.

PROOF. Consider the formal solution (3.2).

- If $\mathcal{S} = \{\emptyset\}$, $\pi(x) = 0$ and we get the degenerate solution $X = 0$.
- If $|\mathcal{S}| = 1$, letting $\mathcal{S} = \{\alpha\}$, we get

$$\pi(x) = -x^{-\alpha} s(\log x), \qquad x > 0.$$

As π must be the Lévy spectral function of some ID random variable X, additional conditions have to be imposed. First the non-negative function $s(\cdot)$ must be bounded. Indeed, $s(\log x) = -x^{\alpha}\pi(x)$; now, as $-x^{\alpha}\pi(x)$ is locally bounded and as $s(x)$ is periodic, necessarily $\sup_{x\in\mathbb{R}} s(x) := \|s\|_\infty < \infty$.

Next, as $\pi(\infty) = 0$, necessarily, α must be a positive number and the restriction $\alpha < 1$ follows from $\int_0^{+\infty} (1 \wedge x)\pi(dx) < \infty$.

Finally, the hazard function $x^{-\alpha}s(\log x)$ should be non-increasing with x. In other words, if we let $s(x) := e^{\alpha\nu(x)}$, for some periodic function ν with periods $\boldsymbol{x} := -\log \boldsymbol{c}$, $\boldsymbol{c} := (c_1, \dots, c_m)$, then $x - \nu(x)$ has to be a non-decreasing function. Finally, it is necessary that $\nu(x)$ be right-continuous. Thus, the pair $(\alpha, s(\cdot))$ has to be admissible.

- If $|\mathcal{S}| = 2$, letting $\mathcal{S} = \{\alpha_1, \alpha_2\}$ with, say $\alpha_1 < \alpha_2$, then we have

$$\pi(x) = -[x^{-\alpha_1} s_1(\log x) + x^{-\alpha_2} s_2(\log x)], \qquad x > 0$$

where $s_l(x) := e^{\alpha_l \nu_l(x)}$, $l = 1,2$ are positive and periodic with the same periods x_i, $i = 1, \dots, m$.

Now, each non-negative function s_l must be bounded; indeed,

$$s_l(\log x) \leq -x^{\alpha_l}\pi(x).$$

As $-x^{\alpha_l}\pi(x)$ is locally bounded and as s_l is periodic, necessarily $\|s_l\|_\infty < \infty$.

Using the same arguments, as for the case $|\mathcal{S}| = 1$, α_1 and α_2 must be in the interval $(0,1)$. Finally, $e^{-\alpha_1 x}s_1(x) + e^{-\alpha_2 x}s_2(x)$ should be non-increasing in such a way that the hazard function $-\pi(x)$ be non-increasing with x.

In other words, if we let $s_l(x) := e^{\alpha_l \nu_l(x)}$, for some periodic functions ν_l with periods $\boldsymbol{x} := -\log \boldsymbol{c}$, $\boldsymbol{c} := (c_1, \dots, c_m)$, then

$$H(x) := e^{-\alpha_1(x - \nu_1(x))} + e^{-\alpha_2(x - \nu_2(x))}$$

should be a non-increasing function.

Let $H_l(x) := e^{-\alpha_l(x-\nu_l(x))}$, $l = 1, 2$, we shall show that *both* $H_1(x)$ and $H_2(x)$ should in fact be non-increasing.

Indeed, let $z_2 > z_1$. It is thus necessary that $H(z_2) \leq H(z_1)$.

Now, as $0 < \alpha_1 < \alpha_2$ and from the boundedness of s_l : $H(z_2) = H_1(z_2)(1 + \epsilon(z_2))$, with $\epsilon(x) = H_2(x)/H_1(x) \underset{x\uparrow+\infty}{\longrightarrow} 0$.

Now, for any $i \in \{1, \ldots, m\}$ and for any $n \in \mathbb{Z}$, the condition $H(z_2) \leq H(z_1)$ also reads

$$\frac{H_1(z_2 + nx_i)}{H_1(z_1 + nx_i)} \leq \frac{1 + \epsilon(z_1 + nx_i)}{1 + \epsilon(z_2 + nx_i)}.$$

But, from the expression of H_1 and the periodicity of $\nu_1(x)$, we get

$$\frac{H_1(z_2 + nx_i)}{H_1(z_1 + nx_i)} = \frac{H_1(z_2)}{H_1(z_1)}.$$

Hence

$$\frac{H_1(z_2)}{H_1(z_1)} \leq \frac{1 + \epsilon(z_1 + nx_i)}{1 + \epsilon(z_2 + nx_i)} \underset{nx_i\uparrow\infty}{\longrightarrow} 1.$$

As a result: $H_1(z_2) \leq H_1(z_1)$. In a similar way, as $nx_i \uparrow -\infty$, one can establish that $H_2(z_2) \leq H_2(z_1)$. Finally, it is necessary that $\nu_l(x)$ be right-continuous. Thus, both pairs $(\alpha_l, s_l(\cdot))$, $l = 1, 2$, have to be admissible. □

Putting all this material together, we obtain

THEOREM 3.1. *Let X be an ID random variable with support $[a, \infty)$, $a > -\infty$. If its LST is solution of the functional equation* (3.1), *then $a = 0$ and its Lévy spectral function reads*

$$\pi(x) = -\sum_{\alpha_l \in \mathcal{S}_1} x^{-\alpha_l} s_l(\log x).$$

Here $\mathcal{S}_1$ is the set of solutions α of $\sum_{i=1}^m \gamma_i c_i^\alpha = 1$ in $(0, 1)$ and for each $\alpha_l \in \mathcal{S}_1$ the pair $(\alpha_l, s_l(\cdot))$ is admissible.

Besides, exploiting the properties of the functions $s_l(\log x)$, the solution depends on the commensurability of the sequence $(\log c_i,\ i = 1, \ldots, m)$. More precisely:

- *If $(\log c_i,\ i = 1, \ldots, m)$ are commensurable with common period $\log c$, then:*
- (i) *$|\mathcal{S}_1| = 1$ and the solution is a semistable distribution*
- (ii) *$|\mathcal{S}_1| = 2$ and the solution is the sum of two independent semistable distributions.*
- *If $(\log c_i,\ i = 1, \ldots, m)$ are noncommensurable, then:*
- (i) *$|\mathcal{S}_1| = 1$ and the solution is a one-sided stable Lévy distribution*
- (ii) *$|\mathcal{S}_1| = 2$ and the solution is the sum of two independent one-sided stable Lévy distributions.*

PROOF. The first part of the theorem is easily obtained by combining Lemmas 3.1 and 3.2. We note that the periodicity condition for $s_l(x)$ is equivalent to

$$s_l(x) = s_l\left(x + \sum_{i=1}^m p_i x_i\right) \tag{3.3}$$

with $x_i = -\log c_i$ and for all $p_i \in \mathbb{Z}$, $i = 1, \dots, m$. Two different cases then arise

- [lattice case]: $x_i = -r_i \log c$, $i = 1, \dots, m$, $c \in (0,1)$, $r_i \in \mathbb{Z}$ and $\gcd(r_i) = 1$. The functions $s_l(x)$ are identified with periodic functions with period $-\log c$.

If $|\mathcal{S}_1| = 1$, we recognize the spectral function of a semistable law. In the case $|\mathcal{S}_1| = 2$, $\varphi_X(p)$ factorizes into the LSTs of two independent random variables (X_1, X_2) where each X_l is an ID random variable with semistable Lévy spectral function

$$\pi_l(x) = -x^{-\alpha_l} s_l(\log x), \qquad x > 0, \ l = 1, 2.$$

- [non-lattice]: the periods x_i are noncommensurable: only the constants $s(x) = s$ are right-continuous bounded solutions of (3.3), since the set $\sum_{i=1}^{m} x_i \mathbb{Z}$ is dense in $\mathbb{R}$. Now, like in the lattice case, the solutions of the functional equation (3.1) are either the one-sided stable Lévy LST ($|\mathcal{S}_1| = 1$) or the product of two such LST ($|\mathcal{S}_1| = 2$). This completes the proof. □

Remark 3.1. Consider the lattice case with common period $-\log c$ defined above. If $\mathcal{S}_1 = \{\alpha\}$, let $\rho > 1$ be defined by $\rho c^{\alpha} = 1$. Equation (3.1) becomes $\varphi_X(p) = \varphi_X^{\rho}(cp)$ which is (2.1). If $\mathcal{S}_1 = \{\alpha_1, \alpha_2\}$, let $\rho_l > 1$ be defined by $\rho_l c^{\alpha_l} = 1$, $l = 1, 2$. The random variable X whose LST solves (3.1) reads $X = X_1 + X_2$ where $\varphi_{X_l}(p) = \varphi_{X_l}^{\rho_l}(cp)$, $l = 1, 2$.

Let us give a simple illustrative example of such a phenomenon.

Example 3.1. Let $m = 2$. Let $c \in (0,1)$ and $c_1 = c$, $c_2 = c^{-1}$, in such a way that $c_1 < 1 < c_2$. Let $\gamma_1 = 1$ and $\gamma_2 := \gamma > 0$. Under these hypothesis, it may be checked that the equation $\tau(\alpha) = 1$ admits two positive solutions if and only if $0 < \gamma < 1/4$ in which case these solutions are

$$0 < \alpha_1 = -\log_c\left(\frac{1 - \sqrt{1 - 4\gamma}}{2\gamma}\right) < \alpha_2 = -\log_c\left(\frac{1 + \sqrt{1 - 4\gamma}}{2\gamma}\right).$$

For $0 < c < \frac{1-\sqrt{1-4\gamma}}{2}$ the solutions α_1 and α_2 belong to the interval $(0,1)$. Thus $X = X_1 + X_2$ where (X_1, X_2) are independent ID random variables whose LSTs are characterized by $\varphi_{X_l}(p) = \varphi_{X_l}^{\rho_l}(cp)$, $l = 1, 2$ with $(\rho_1 > 1, \rho_2 > 1)$ defined by

$$\rho_1 = c^{-\alpha_1} = \frac{1 - \sqrt{1 - 4\gamma}}{2\gamma} \quad \text{and} \quad \rho_2 = c^{-\alpha_2} = \frac{1 + \sqrt{1 - 4\gamma}}{2\gamma}.$$

As it was emphasized in Theorem 3.1 the number of solutions to $\tau(\alpha) = 1$ in the interval $(0,1)$ is a central point in characterizing the solutions of the functional equation (3.1).

We now come to solution explicit form of (3.1), exploiting a correspondence between LST of ID random variable and its Lévy spectral function (2.2).

PROPOSITION 3.1. *Let X be a GSS random variable with Lévy spectral function*

$$\pi(x) = -\sum_{\alpha_l \in \mathcal{S}_1} x^{-\alpha_l} s_l(\log x) \tag{3.4}$$

where $\mathcal{S}_1$ is the set of solutions α of $\sum_{i=1}^m \gamma_i c_i^\alpha = 1$ in $(0,1)$ and such that for each $\alpha_l \in \mathcal{S}_1$ the pair $(\alpha_l, s_l(\cdot))$ is admissible. Assume in addition that the $s_l(\cdot)$ are continuous with left and right derivatives at each point. Two cases arise:

- *If $(\log c_i,\ i = 1, \dots, m)$ are noncommensurable, then the functions $s_l(\cdot)$ are constant with $s_l := s_l(\cdot)$. We have*

$$\varphi_X(p) = \exp\left\{ - \sum_{\alpha_l \in \mathcal{S}_1} \tilde{s}_l p^{\alpha_l} \right\}, \qquad p \geq 0$$

with $\tilde{s}_l = s_l \Gamma(1-\alpha)$.

- *If $(\log c_i,\ i = 1, \dots, m)$ are commensurable with common period $\log c$, then the functions $s_l(\cdot)$ are periodic with period $\log c$, with, say, $(s_{n,l})_{n\in\mathbb{Z}}$ as Fourier series coefficients. Then, it holds*

$$\varphi_X(p) = \exp\left\{ - \sum_{\alpha_l \in \mathcal{S}_1} p^{\alpha_l} \tilde{s}_l(\log p) \right\}, \qquad p \geq 0 \tag{3.5}$$

where $\tilde{s}_l$ are non-negative periodic functions given by their Fourier series coefficients $\tilde{s}_{n,l} := s_{-n,l}\Gamma(1 - \alpha_l + \frac{2i\pi n}{\log c})$, $n \in \mathbb{Z}$.

PROOF. The first part follows from the identity

$$p^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1 - e^{-px}) x^{-(1+\alpha)} dx. \tag{3.6}$$

Concerning the second part, for $l = 1, 2$, let

$$s_l(\log x) = \sum_{n\in\mathbb{Z}} s_{n,l} e^{-2i\pi n \log x / \log c} \tag{3.7}$$

with $s_{n,l}$, $n \in \mathbb{Z}$, the Fourier coefficients of $s_l(\cdot)$. Combining (3.4), (3.7), we obtain

$$\pi(x) = - \sum_{\alpha_l \in \mathcal{S}_1} \sum_{n\in\mathbb{Z}} s_{n,l} x^{-(\alpha_l + 2i\pi n/\log c)}. \tag{3.8}$$

Thus, using (3.6) with $\alpha \in \mathbb{C}$, $0 < \mathrm{Re}(\alpha) < 1$, (3.8) gives

$$\int_0^{+\infty} (1 - e^{-px}) d\pi(x) = \sum_{\alpha_l \in \mathcal{S}_1} \sum_{n\in\mathbb{Z}} s_{n,l} \Gamma\left(1 - \alpha_l - \frac{2i\pi n}{\log c}\right) p^{\alpha_l + 2i\pi n/\log c}.$$

From (3.5), we obtain

$$\tilde{s}_l(\log p) = \sum_{n\in\mathbb{Z}} \tilde{s}_{n,l} e^{-2i\pi n \log p / \log c}$$

where

$$\tilde{s}_{n,l} := s_{-n,l} \Gamma\left(1 - \alpha_l + \frac{2i\pi n}{\log c}\right), \qquad n \in \mathbb{Z},\ l = 1, 2.$$

This completes the proof. □

Remark 3.2. Given any non-negative periodic function $\tilde{s}(\cdot)$ and $\alpha \in (0,1)$, the function

$$\exp\{-p^{\alpha}\tilde{s}(\log p)\}$$

is not the LST of a ID random variable with support $[0,\infty)$. It is the case if and only if $p^{\alpha}\tilde{s}(\log p)$ has a completely monotone derivative (see Feller (1971), p. 450). The functions $\tilde{s}(\cdot)$ defined in the preceding proposition obviously fulfill this condition by construction.

In the forthcoming section, we shall need the following definition.

DEFINITION 3.2. For a given $\alpha \in \mathbb{R}$ and a positive function $\tilde{s}(\cdot)$, the pair $(\alpha, \tilde{s}(\cdot))$ will be called a Laplace-admissible pair, if we have

(i) $\alpha \in (0,1)$,

(ii) function $\tilde{s}(\cdot)$ is a non-negative periodic function with periods $\boldsymbol{x} := -\log \boldsymbol{c}$, $\boldsymbol{c} := (c_1, \ldots, c_m)$

(iii) $p^{\alpha}\tilde{s}(\log p)$ has a completely monotone derivative.

3.2 *The shifted generalized semistable laws*

Let $\beta_i \in \mathbb{R}$, $i = 1, \ldots, m$ and consider the functional equation

$$\widetilde{\varphi}(p) = \prod_{i=1}^{m} e^{-p\beta_i}\widetilde{\varphi}^{\gamma_i}(c_i p), \qquad p \geq 0, \tag{3.9}$$

in the class of LST of random variables with support $[a,\infty)$ for some $a > -\infty$. With $\beta := \sum_{i=1}^{m}\beta_i$, this is also

$$\widetilde{\varphi}(p) = e^{-p\beta}\prod_{i=1}^{m}\widetilde{\varphi}^{\gamma_i}(c_i p), \qquad p \geq 0. \tag{3.10}$$

PROPOSITION 3.2. *Let X a random variable whose LST is solution to* (1.1). *The LST of $\widetilde{X} = X + x$ is solution of* (3.9) *or equivalently* (3.10) *with $\beta = x(1 - \sum_{i=1}^{m}\gamma_i c_i)$. Conversely, let $\widetilde{X}$ be an ID random variable with support $[x,\infty)$, $x > -\infty$ whose LST is solution of the functional equation* (3.9) *or equivalently* (3.10), *then $x = \beta/(1-\sum_{i=1}^{m}\gamma_i c_i)$ and $\widetilde{X} = X + x$ where the LST of X is solution to* (1.1).

PROOF. Consider the LST $\psi(p) := e^{-px}\varphi(p)$ where $\varphi(p)$ is the LST of the random variable X which solves (1.1). Then clearly $\psi(p)$ solves (3.10) or (3.9) with $\beta = x(1 - \sum_{i=1}^{m}\gamma_i c_i)$ and $\psi(p)$ is the LST of the shifted random variable $\widetilde{X} = X + x$.

Conversely, suppose there exists a LST $\widetilde{\varphi}(p)$ in the class of LST of random variables with support $[x,\infty)$ for some $x > -\infty$ which solves (3.10) or (3.9).

Introduce $\varphi(p) := \exp(\beta/(1 - \sum_{i=1}^{m}\gamma_i c_i))\widetilde{\varphi}(p)$. Then $\varphi(p)$ satisfies (1.1) with support $[x - \beta/(1 - \sum_{i=1}^{m}\gamma_i c_i), \infty)$ and from Theorem 3.1, $x = \beta/(1 - \sum_{i=1}^{m}\gamma_i c_i)$. □

Remark 3.3. (i) We note that in (3.9) the β_i, $i = 1, \ldots, m$ are not unique and that its solution is related to $\beta := \sum_{i=1}^{m}\beta_i$.

(ii) Shifting by x the solution (3.5) of (1.1), solutions of (3.10), are found to be

$$\varphi_{\tilde{X}}(p) = \exp\left\{-\left[px + \sum_{\alpha_l \in \mathcal{S}_1} p^{\alpha_l} \tilde{s}_l(\log p)\right]\right\}, \qquad p \geq 0.$$

(iii) Let X be a random variable whose LST fulfills (3.1), not necessarily ID. From the Lau-Rao-Shanbhag theorem, one can obtain a formal solution under the form $\exp\{-\sum_{\alpha_l \in \mathcal{S}} p^{\alpha_l} \tilde{s}_l(\log p)\}$, $p \geq 0$, where $\mathcal{S}$ are solutions to $\tau(q) = 1$ (with at most two solutions $\alpha_1 < \alpha_2$) and $\tilde{s}_l$ are positive, bounded and periodic functions. When $|\mathcal{S}| = 1$, $\varphi_X(p)$ is the LST of a semistable distribution, hence necessarily ID. When $|\mathcal{S}| = 2$, we could not prove that solutions to (3.1) are necessarily ID.

3.3 *Properties of GSS laws*

Let us now stress some additional properties of GSS random variables with support $[0, \infty)$.

1/ Semi-selfsimilarity:

One-sided Lévy-stable distributions are interesting in practice because of the self-similarity property of the associated subordinator process with stationary independent increments. Indeed, the induced strictly Lévy-stable processes $\{X(t), t \geq 0\}$ share the strict selfsimilarity property: for any $\varsigma > 0$

$$\{X(\varsigma t), t \geq 0\} \stackrel{d}{=} \{\varsigma^{1/\alpha} X(t), t \geq 0\}. \tag{3.11}$$

If this holds, $\{X(t), t \geq 0\}$ is said to be strictly selfsimilar with characteristic exponent $1/\alpha$.

For strictly semistable random variables, their associated strictly Lévy-semistable processes satisfy $\{X(t), t \geq 0\} \stackrel{d}{=} \{cX(\rho t), t \geq 0\}$, for some $\rho > 1$ and c related through $\rho c^\alpha = 1$ with $\alpha \in (0, 1)$.

Thus, for some $\rho > 1$, $\{X(\rho t), t \geq 0\} \stackrel{d}{=} \{\rho^{1/\alpha} X(t), t \geq 0\}$, which is also: for some $\rho > 1$ and for any $n \in \mathbb{Z}$

$$\{X(\rho^n t), t \geq 0\} \stackrel{d}{=} \{\rho^{n/\alpha} X(t), t \geq 0\}.$$

Such random processes are said to be semi-selfsimilar (with exponent $1/\alpha$) as (3.11) only holds for those ς of the particular form $\varsigma = \rho^n$, $n \in \mathbb{Z}$, concentrating at zero (see Sato (1999)).

Note that the process interpretation of functional equation (3.1) reads

$$\{X(t), t \geq 0\} \stackrel{d}{=} \left\{\sum_{i=1}^{m} c_i X_{(i)}(\gamma_i t), t \geq 0\right\}$$

where $\{X_{(i)}(t), t \geq 0\}$ are m-iid copies of Lévy processes satisfying $X_{(i)}(1) \stackrel{d}{=} X$.

If $|\mathcal{S}_1| = 1$, in the lattice (non-lattice) case, the solution to the functional equation (3.1) for X is semistable (stable). As a result, the process associated with such GSS laws is semi-selfsimilar (selfsimilar).

If $|\mathcal{S}_1| = 2$, in the lattice (non-lattice) case, GSS distributions are the convolution of two semi-selfsimilar (selfsimilar) laws. This constitutes a further extension of the semi-selfsimilarity (selfsimilarity) notion for the associated subordinator process. In this case indeed, such process has two characteristic exponents.

2/ If $|\mathcal{S}_1| = 2$, (3.5) yields:

$$\varphi_X(p) = \exp\{-[p^{\alpha_1}\tilde{s}_1(\log p) + p^{\alpha_2}\tilde{s}_2(\log p)]\}, \qquad p \geq 0,\ \alpha_1 < \alpha_2$$

with two non-negative bounded periodic functions, $\tilde{s}_1$ and $\tilde{s}_2$. Clearly, we have

$$1 - \varphi_X(p) \underset{p\downarrow 0}{\sim} p^{\alpha_1}\tilde{s}_1(\log p).$$

Thus X is "close" to be regularly-varying with tail index $\alpha_1 > 0$, just like the Lévy stable law was. In fact, although $L(p) := \tilde{s}_1(\log p)$ is not slowly varying, it satisfies the weaker condition that, for all $t > 0$, $L(tp)/L(p)$ has a liminf and a limsup for small p. By Tauberian theorem (see Feller (1971), p. 445), X is heavy-tailed, with characteristic exponent α_1.

4. GSS distributions as limit laws in Statistics under random geometric growth

In this section, we show that generalized semistable distributions may be seen as limit laws for some renormalized sum of an iid sample. It should be emphasized that we have here no pretention of fully characterizing the attraction basin of GSS laws. Some work in this direction for the particular case of semistable distributions may be found in Grinevich and Khokhlov (1993), Kruglov (1972), Shimizu (1970).

4.1 *The semistable case*: *Preliminaries*

Consider a particular lattice GSS model with $\gamma_i = 1$, $c_i = c^{r_i}$, $r_i \in \mathbb{N}$, $i = 1, \ldots, m$, $\gcd(r_i) = 1$ and $\sum c_i < 1$. This guarantees that we are in a semistable case with a unique $\alpha \in (0,1)$ defined by $\tau(\alpha) = 1$.

Define now the integer-valued function $n(r)$ of $r \in \mathbb{N}$, recursively by

$$n(r) = \mathbf{1}(r^* > r) + \sum_{i=1}^{m} n(r - r_i)\mathbf{1}(r_i \leq r), \qquad r \in \mathbb{N}^*,\ n(0) = 1$$

with $r^* := \max_{i=1,\ldots,m} r_i$. This sequence is a deterministic multitype branching process which states that the number of individuals at discrete time r is obtained as follows: at time $r = 0$, a single ancestor is available; this ancestor gives birth to m first generation sons as a whole, a type-i son coming to life at time $r_i > 0$. The ancestor dies at time r^* when it gives birth to its last son. Each first generation son repeats the same splitting program, starting from its birth time, and so forth for the subsequent generations. This construction simply is a deterministic version of the age-dependent Crump-Mode branching process. Clearly, under these hypothesis, $n(r) \sim_{r\uparrow\infty} K\rho^r$ for some $K > 0$, where $\rho > 1$ is uniquely defined by $\sum_{i=1}^{m} \rho^{-r_i} = 1$. Recalling that the condition $\tau(\alpha) = 1$ reads

$$\sum_{i=1}^{m} c_i^{\alpha} = \sum_{i=1}^{m} c^{\alpha r_i} = 1,$$

we conclude that $\rho c^{\alpha} = 1$.

Let $(\mathcal{X}_j, j \geq 1)$ be a sequence of iid random variables, distributed like a positive random variable $\mathcal{X}$, with LST ψ. Let $Z(r) := \sum_{j=1}^{n(r)} \mathcal{X}_j$. With these preliminaries, we have

PROPOSITION 4.1. *Let $\tilde{s}(\cdot)$ some Laplace-admissible scale function and $L(\cdot)$ some slowly varying function at infinity. If for small p we have*

$$1-\psi(p) \underset{p\downarrow 0+}{\sim} \frac{1}{K}[pL(p^{-1})]^{\alpha}\tilde{s}(\log(pL(p^{-1})))$$

then there exists $\sigma(r)>0$ such that the rescaled process $\widetilde{Z}(r) := Z(r)/\sigma(r)$, $r\in\mathbb{N}$, converges in law, as $r\uparrow\infty$, to some semistable law.

PROOF. First, define $\sigma(r)$ by $L(\sigma(r))/\sigma(r) = c^r$. Next, consider $\widetilde{\varphi}_r(p) := \boldsymbol{E}e^{-p\tilde{Z}(r)}$, under our hypothesis with $\varepsilon_2(r) \underset{r\uparrow\infty}{\to} 0$, for large r we have

$$\begin{aligned}
\widetilde{\varphi}_r(p) &= \psi\left(\frac{p}{\sigma(r)}\right)^{n(r)} = \exp\{-n(r)[1-\psi(p/\sigma(r))][1+\varepsilon_2(r)]\} \\
&\sim \exp\left\{-\frac{n(r)}{K}\left[\frac{p}{\sigma(r)}L\left(\frac{\sigma(r)}{p}\right)\right]^{\alpha}\tilde{s}\left(\log\left[\frac{p}{\sigma(r)}L\left(\frac{\sigma(r)}{p}\right)\right]\right)\right\} \\
&\sim \exp\left\{-\frac{n(r)}{K}\left[\frac{L(\sigma(r))}{\sigma(r)}\right]^{\alpha}p^{\alpha}\tilde{s}\left(\log\left[p\frac{L(\sigma(r))}{\sigma(r)}\right]\right)\right\} \\
&\sim \exp\{-p^{\alpha}\tilde{s}(\log[pc^r])\}.
\end{aligned}$$

As $r\in\mathbb{N}$, if $\tilde{s}(\log p)$ has period $\log c$, it follows that

$$\widetilde{\varphi}_r(p) \underset{r\uparrow\infty}{\to} e^{-p^{\alpha}\tilde{s}(\log p)}$$

which is the LST of some semistable random variable X, as required. □

We now show how to obtain lattice GSS laws as the weak limit of some rescaled sum of a Poisson random number of iid random variables when the Poisson intensity has geometric growth.

4.2 *The full GSS case*

In the GSS model, let us now simply assume that $c_i,\gamma_i>0$, $i=1,\dots,m$ are such that $\mathcal{S}_1=\{\alpha\in(0,1):\tau(\alpha)=1\}$ is not empty. In the lattice case, there exists $c\in(0,1)$, $r_i\in\mathbb{Z}$ such that $c_i=c^{r_i}$, $i=1,\dots,m$, with $\gcd(|r_i|)=1$.

Let now $N(r)$ be a Poisson process with intensity $\lambda(r)$ satisfying the functional equation

$$\lambda(r)=\sum_{i=1}^{m}\gamma_i\lambda(r-r_i), \qquad \forall r\in\mathbb{Z}.$$

Solutions of the type $\lambda(r)=K\rho^r$ with $K>0$ and $\rho>0$, exist if there exists $\rho>0$ satisfying $\sum_{i=1}^{m}\gamma_i\rho^{-r_i}=1$. Recalling that condition $\tau(\alpha)=1$ reads

$$\sum_{i=1}^{m}\gamma_i c_i^{\alpha}=\sum_{i=1}^{m}\gamma_i c^{\alpha r_i}=1, \qquad \text{for each} \quad \alpha\in\mathcal{S}_1,$$

we can take $\rho=\rho_\alpha$ where, for each $\alpha\in\mathcal{S}_1$, $\rho_\alpha c^\alpha=1$ and $\rho_\alpha>1$.

For each $\alpha\in\mathcal{S}_1$, we will denote by $N_\alpha(r)$ the Poisson process with intensity $\lambda_\alpha(r)=K_\alpha\rho_\alpha^r$, $K_\alpha>0$. Furthermore, $\lambda(r)=\sum_{\alpha\in\mathcal{S}_1}\lambda_\alpha(r)$. If $|\mathcal{S}_1|=2$, we assume these two

Poisson processes to be independent so that $N(r) = \sum_{\alpha \in \mathcal{S}_1} N_\alpha(r)$ is a Poisson process. For each $\alpha \in \mathcal{S}_1$, let $(\mathcal{X}_\alpha \stackrel{d}{=} \mathcal{X}_{\alpha,j}, j \geq 1)$ be an iid sequence, independent of $N_\alpha(r)$ and let $\psi_\alpha(p)$ be the LST of $\mathcal{X}_\alpha$. If $|\mathcal{S}_1| = 2$, assume $(\mathcal{X}_{\alpha,j}, j \geq 1, N_\alpha(r))$, $\alpha \in \mathcal{S}_1$, to be mutually independent. Consider now the sum processes

$$Z_\alpha(r) := \sum_{j=1}^{N_\alpha(r)} \mathcal{X}_{\alpha,j} \quad \text{and} \quad Z(r) := \sum_{\alpha \in \mathcal{S}_1} Z_\alpha(r).$$

Clearly, these processes are infinitely divisible as Poisson sums of iid random variables. Observe that $Z(r) \stackrel{d}{=} \sum_{j=1}^{N(r)} \mathcal{X}_j(r)$, with $(\mathcal{X}(r) \stackrel{d}{=} \mathcal{X}_j(r), j \geq 1)$ an iid sample for each r defined by the Bernoulli mixture: $\mathcal{X}(r) \stackrel{d}{=} B_r \mathcal{X}_{\alpha_1,j} + (1 - B_r)\mathcal{X}_{\alpha_2,j}$ with $\mathcal{S}_1 = \{\alpha_1, \alpha_2\}$ and $B_r \in \{0,1\}$ a Bernoulli variable, independent of $\mathcal{X}_{\alpha,j}$, $\alpha \in \mathcal{S}_1$, for which $\boldsymbol{P}(B_r = 1) = \lambda_{\alpha_1}(r) / \sum_{\alpha \in \mathcal{S}_1} \lambda_\alpha(r)$.

PROPOSITION 4.2. *Assume*

$$1 - \psi_\alpha(p) \underset{p \downarrow 0+}{\sim} \frac{1}{K_\alpha} [pL(p^{-1})]^\alpha \tilde{s}_\alpha(\log(pL(p^{-1}))), \qquad \alpha \in \mathcal{S}_1$$

for some Laplace-admissible scale function $\tilde{s}_\alpha(\cdot)$. Then, there exists $\sigma(r) > 0$ such that the rescaled process

$$\widetilde{Z}(r) := Z(r)/\sigma(r)$$

converges in law to some GSS random variable X as $r \uparrow \infty$.

PROOF. Define $\sigma_\alpha(r)$ by $[\sigma_\alpha(r)/L(\sigma_\alpha(r))]^{-\alpha} = 1/\rho_\alpha^r$, $\alpha \in \mathcal{S}_1$. With $\widetilde{\varphi}_r(p) := \boldsymbol{E} e^{-p\widetilde{Z}(r)}$ and from the Poisson structure, we get

$$\widetilde{\varphi}_r(p) = \exp\left\{ - \sum_{\alpha \in \mathcal{S}_1} \lambda_\alpha(r)[1 - \psi_\alpha(p/\sigma_\alpha(r))] \right\} \sim e^{-p^\alpha \tilde{s}(\log(p\rho^{-r/\alpha}))}.$$

Recalling that $\rho_\alpha c^\alpha = 1$ for each α, we observe that $\sigma_\alpha(r) := \sigma(r)$ is independent of α with $[\sigma(r)/L(\sigma(r))] = c^{-r}$ defining $\sigma(r)$. Next,

$$\lambda_\alpha(r)[1 - \psi_\alpha(p/\sigma(r))] \sim p^\alpha \tilde{s}_\alpha(\log(pL(\sigma(r))/\sigma(r)))$$

and as function $\tilde{s}_\alpha(\cdot)$ has period $\log c := -\frac{1}{\alpha} \log \rho$ and

$$\widetilde{\varphi}_r(p) \underset{r \uparrow \infty}{\longrightarrow} e^{-\sum_{\alpha \in \mathcal{S}_1} p^\alpha \tilde{s}_\alpha(\log p)}$$

which is the LST of some GSS random variable X, as required. □

5. Conclusions

A functional equation generalizing the one characterizing semistable distributions has been considered and solved. GSS laws have been shown to be limiting laws for normalized and centered sums of specific iid random variables when the sample size (either deterministic or random) grows geometrically.

Further extensions of these GSS distributions are presently under study, namely the ones obtained when randomizing the integer m, and/or the location-scale and intensity parameters $((\beta_i, c_i, \gamma_i)\ i = 1, \ldots, m)$ in (1.1), following the works of Guivarc'h (1990), Kahane and Peyrière (1976), Liu (1997), Mandelbrot (1974*a*, 1974*b*) and Shimizu and Davies (1981).

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