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On *Lévy–Fréchet* processes and related self-similar and semi-self-similar ones

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Abstract

Lévy (semi-) stable processes are (semi-) self-similar and, as such, have recently drawn attention of many researchers. On the other hand, there are lots of interesting (semi-) self-similar processes that are not in the *Lévy* class. Here, we focus on classes of *Markov* processes related to the extremal processes of *Lévy* (semi-) stable ones that are all (semi-) self-similar. These are instructive for those working in applied fields.

1. Introduction and physical motivations

The statistical notions of stability and self-similarity have been introduced by *Paul Lévy* in the 1940s. They have been recognized as important ones in Physics and related fields more recently, starting from the 1970s (see [1,4,14,30,34,42,44,45,47,48] and references therein).

We first recall these notions of interest both from the theoretical and practical points of view.

Stability

Here we recall some salient facts concerning one-sided (i.e. with positive support) stable random variables (see [43,48] for a recent overview of these problems). The adjective stable refers to the well-known identity in distribution

$$\text{for any } c_1 > 0,\ c_2 > 0,\ \text{there exists } c > 0 : c_1X_1 + c_2X_2 \stackrel{d}{=} cX,$$

with $X \stackrel{d}{=} X_1 \stackrel{d}{=} X_2$ mutually independent (with symbol $\stackrel{d}{=}$ standing for identity in law). These models are known [3,8,43] to belong to a subclass of infinitely divisible (ID) random variables which are those for which the nth root of their characteristic function still is a characteristic function for any $n \geqslant 1$ [46]. This notion is therefore closely related to the one of *Lévy* processes, that is with stationary independent increments (sii).

Besides, stable distributions derive their importance from the fact that they are the limit laws for sums of independent and identically distributed (iid) random variables, say χ_m, $m \geqslant 1$, after a convenient location-scale transform, which can be found in [48], for example, as well as striking motivations in Physics.

Similarly, there exists a concept of max-stability, substituting the symbol max to + in the above identity (see [15], for example).

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Self-similarity

A stochastic process $X(t)$, $t \geqslant 0$, is said to be self-similar if for any $a > 0$, there exists $b > 0$ such that

$$\{X(at), t \geqslant 0\} \overset{d}{=} \{bX(t), t \geqslant 0\}$$

meaning the identity in law for any finite-dimensional distributions.

Besides the stability property, one-sided *Lévy*-stable processes are interesting in practice because they are the only ones in the *Lévy* class to possess the self-similarity property. Indeed, as is well-known, there exists an $\alpha \in (0,1)$ such that the associated *Lévy*-stable process $X(t)$ shares the additional self-similarity property: for any $t > 0$

$$\{X(t), t \geqslant 0\} \overset{d}{=} \{t^{1/\alpha}X, t \geqslant 0\}.$$

Here $X \overset{d}{=} X(1)$ is a random variable. With $\mathbf{E}$ denoting mathematical expectation, we shall let $\varphi(\lambda) := \mathbf{E}\,e^{-\lambda X}$ denote its *Laplace Stieltjes* transform (LST). For $t > 0$, the variable $X(t)$ is defined as the variable whose LST is $\varphi_{X(t)}(\lambda) = \varphi(\lambda)^t$, raising φ to the power t. This property is the one of infinite divisibility of X. The uni-dimensional version of the above identity in law can then be interpreted as

$$\varphi(\lambda) = \varphi(t^{-1/\alpha}\lambda)^t \quad \text{for any } t > 0.$$

Semi-stability

We now discuss a concept whose generality is larger than the one of stability, namely the one of semi-stability.

A one-sided semi-stable variable X is identified with one whose LST satisfies a functional equation of the form

$$\varphi(\lambda) = \varphi(c\lambda)^\gamma, \quad \lambda \geqslant 0,$$

for some $c > 0$, $\gamma > 1$. This is a scale-invariance property.

This notion of semi-stability was first introduced by *Paul Lévy* in 1937 [22] (see [24, p. 45] for a survey on this point). Additional references on these topics are [2,25–27]. The book [43] is the most instructive survey. Note that stable variables are semi-stable.

Substituting the probability distribution function to the LST in the above functional equation, one obtains the class of max-semi-stable variables (see [15], for example).

There are analogies between the concept of semi-stability (as a scale-invariance concept) and the one discussed in [47] on discrete scale-invariance arising from Renormalization Group theory in Physics (see [16]).

One-sided semi-stable variables are thus ID; they contain the class of stable laws. *Lévy* semi-stable processes can easily be defined accordingly.

Besides, semi-stable variables constitute a subclass of semi-self-decomposable (itself a sub-class of ID ones) variables in the following sense [43, p. 90]. A random variable X is said to be semi-self-decomposable if, for some $c \in (0,1)$

$$X \overset{d}{=} cX_1 + R,$$

where R is an ID positive random variable, independent of cX_1, with $X \overset{d}{=} X_1$ (note that a stable distribution is self-decomposable in that the above identity in law holds for *any* such c, with $R \overset{d}{=} (1 - c^\alpha)^{1/\alpha}X_2$ and $X \overset{d}{=} X_2$).

Semi-self-similarity

A stochastic process $X(t)$, $t \geqslant 0$, is said to be semi-self-similar if there exist $a \neq 1$ and $b > 0$ such that

$$\{X(at), t \geqslant 0\} \overset{d}{=} \{bX(t), t \geqslant 0\}.$$

Note that a self-similar process is semi-self-similar.

Besides the semi-stability property, one-sided *Lévy* semi-stable processes are interesting in practice because they are the only ones in the *Lévy* class to be semi-self-similar. Indeed, there exist $\gamma > 1$ and $\alpha \in (0,1)$ such that the associated *Lévy* semi-stable process $X(t)$ shares the additional semi-self-similarity property: for any $t > 0$

$$\{X(\gamma t), t \geqslant 0\} \overset{d}{=} \{\gamma^{1/\alpha}X(t), t \geqslant 0\}.$$

It should be clear that, although the notions of (semi-) stability fit with the one of (semi-) self-similarity in the case of *Lévy* processes, the latter are much more general than the former. To take a famous example, the fractional *Brownian* motion designed by Mandelbrot et al. and followers (see [7,14,29,30,41]) is well-known to be self-similar although not even in the *Markov* processes' class.

We now come to the class of processes of interest in this manuscript which will serve as an additional illustration of the previous remark. These are all processes of concrete interest, in the *Markov* class, that are (semi-) self-similar. They are related to extremal processes of (semi-) stable *Lévy* processes, as their largest positive jump process over time.

We start with motivations on extremal processes related to compound *Poisson* processes. Although these are never (semi-) self-similar, they will serve as a simple illustration of the extremal processes we wish to consider. Besides, they will be used as weak approximations to design the announced models with the desired properties.

In many domains of applied physics and engineering, the space-time phenomenon to be studied presents itself as a sequence of events with random amplitude occurring randomly over time. This can be for example a sequence of earthquake magnitude over some region [13,33,49] or the sequence of damages met by the customers of some insurance company in Finance [6], or alternatively the random users' demands for network or energy resources in Telecommunications' or power supply management technology (see [6,23], for precise motivations and results in both Physics and Finance).

In Section 2, the following simple statistical model for such sequences is considered: events of random iid amplitude, say $(\chi; \chi_m,\ m \geqslant 1)$, occur randomly in time according to a *Poisson* process, say $N(t)$, with intensity mt, $m > 0$. This process, say $\Delta(t)$, is a continuous-time process called the jump (or amplitude) process which at time $t \geqslant 0$ is in state $\chi_{N(t)}$ if $N(t) > 0$, x_0 otherwise, with x_0 the left-endpoint of the support of χ's distribution.

Upon cumulating these magnitudes over time, we are left with a simple compound *Poisson* process $X(t)$ which integrates the previous jump process.

In this context, the largest magnitude process (or largest jump extremal process) obtained while considering the maximum amplitude over past records is of particular interest. Also, the time to failure process which is the first time at which some magnitude will cross some pre-assigned magnitude level deserves special attention; exchanging the roles of space (magnitude) and time yields an "inverse" extremal process which interprets as the time at which the jump process first crosses clock time. The purpose of Section 2 is to derive the structure of both extremal and inverse extremal processes and to study some of their interesting properties, under the announced hypothesis for the jump process. It turns out that the first process has a *Markov* structure, whereas the second one is a process with independent but unstationary increments.

In many of the examples of interest discussed above, an additional characteristic feature of the phenomenon under study is the following: events of tiny amplitude (say larger than but close to some threshold $\varepsilon > 0$, with small ε) are extremely numerous or frequent.

To take an example, an important feature of earthquake catalogs is that, although data on small earthquakes are strongly deficient, due to incomplete, bad registration of low magnitude events, these are certainly extremely numerous. One way to address this problem would have been to consider the data as a realization of a truncated distribution; in effect, this amounts to the assumption that there exists some fixed detection threshold above which all earthquakes are recorded. Such a threshold could have been either estimated from the data or deduced from physical considerations, at the price of discarding part of the data. Lowering this threshold, a huge amount of small events will certainly emerge.

On the other hand, these are punctuated with some rare events but with comparatively very large macroscopic amplitude (the ones of interest to the Engineer): the physical image is therefore the one of bursts of activity immersed in an ocean of "insignificant" microevents (a coarse version of *Lévy* white noise).

We first investigate properties of such models when the amplitude, say χ^ε, follows a heavy-tailed [1,44] *Pareto* distribution of parameter α over $[\varepsilon, \infty)$ of the form

$$\Pr(\chi^\varepsilon > x) = (x/\varepsilon)^{-\alpha}, \quad x \geqslant \varepsilon,\ \alpha > 0, \tag{1.1}$$

and when the underlying *Poisson* process has intensity $\varepsilon^{-\alpha}t$ (traducing the desired properties).

When $\alpha \in (0, 1)$, performing the *Poisson* sum of such iid magnitude's sequence, and passing to the limit $\varepsilon \downarrow 0$ yield the celebrated one-sided *Lévy*-stable process.

When $\alpha > 0$, computing the *Poisson* maximum of such iid magnitude processes, and passing to the limit $\varepsilon \downarrow 0$ yield a process to be identified with the *Fréchet* extremal process. Some properties (essentially the dependence structure) of this limiting process and of the associated inverse *Fréchet* process are supplied, exploiting the constructions of Section 2. Related processes such as *Gumbel* extremal processes are also briefly investigated. It turns out that all these limiting processes are self-similar, illustrating the fact that self-similarity may be compatible with different processes' *Markov* structures. All this is the purpose of Section 3.

In Section 4, a related class of self-similar processes is next studied in some detail, namely the class of Geometric *Lévy* and *Fréchet* processes. They are obtained while subordinating the standard *Lévy*, *Fréchet*-stable processes in self-similar Exponential time.

It turns out that these limiting Geometric processes (*Lévy*, *Fréchet*) are also self-similar.

In Section 5, we consider a *Lévy* semi-stable process, extending the previous construction. His extremal and inverse extremal processes are studied. These may be obtained, as before, as limiting processes when the amplitude, say χ^{ε}, follows now a distribution over $[\varepsilon, \infty)$ of the more general form

$$\Pr(\chi^{\varepsilon} > x) = \frac{1}{m_{\varepsilon}} x^{-\alpha} \exp \alpha v(\log x), \quad x \geqslant \varepsilon, \tag{1.2}$$

for some normalization constant $m_{\varepsilon} > 0$. Here, for some constant $c > 0$, v is a bounded periodic function with period $-\log c$ on the real line. It turns out that all these limiting processes are semi-self-similar.

In Section 6, a related class of semi-self-similar processes is finally investigated. They are the Geometric *Lévy* semi-stable and Geometric *Fréchet* max-semi-stable processes. These semi-self-similar processes are obtained while subordinating the *Lévy*, *Fréchet* semi-self-similar processes in Exponential time. In the sequel, we will speak of *Fréchet* semi-stable processes in place of *Fréchet* max-semi-stable processes.

2. Largest jump process of a compound *Poisson* process and its inverse

In this section, we supply the construction of the largest jump (extremal) and time to failure processes when the jumps' sum process is a standard compound *Poisson* process. Some of their statistical properties are supplied. Similar considerations and physical motivations into this problem can be found in [23] and the references therein.

2.1. Largest jump extremal process

Assume $(X(t), t \geqslant 0)$ is a compound *Poisson* process hence with random stationary independent increments, say χ; we let $x_0 := \inf(x : \Pr(\chi \leqslant x) > 0)$ be the bottom value (or left endpoint) of the support of χ's distribution. We assume in the sequel that $\Pr(\chi \leqslant x)$ is a continuous function of x. Then, from the *Poisson* structure

$$\mathbf{E}\, \mathrm{e}^{-\lambda X(t)} = \mathrm{e}^{-mt(1-\mathbf{E}\,\mathrm{e}^{-\lambda\chi})} \tag{2.1}$$

with $\mathbf{E}\,\mathrm{e}^{-\lambda\chi}$ the LST of the increment random variable (the jump height χ). It is well-known that $X(t)$ is a *Poisson* sum of its increments

$$X(t) = 0 \cdot \mathbf{1}(N(t) = 0) + \left[\sum_{m=1}^{N(t)} \chi_m\right] \cdot \mathbf{1}(N(t) > 0), \quad t \geqslant 0, \tag{2.2}$$

where $N(t)$, $t \geqslant 0$, is a standard *Poisson* process with intensity $\mathbf{E}N(t) = mt$, $m > 0$.

Next, we consider the continuous-time pure jump process

$$\Delta(t) := x_0 \cdot \mathbf{1}(N(t) = 0) + [\chi_{N(t)}] \cdot \mathbf{1}(N(t) > 0), \quad t \geqslant 0, \tag{2.3}$$

which is the jump process associated to the cumulative jump process $X(t)$, $t \geqslant 0$. We let

$$X_*(t) := \max_{s \leqslant t} \Delta(s) \tag{2.4}$$

be the associated largest jump (or extremal) process. Clearly,

$$X_*(t) = \max(x_0, \chi_1, \ldots, \chi_{N(t)}) \tag{2.5}$$

and, with $x \geqslant x_0$

$$\Pr(X_*(t) \leqslant x) = \mathbf{E}\,\mathrm{e}^{-\lambda N(t)} \mid_{\lambda = -\log \Pr(\chi \leqslant x)} = \mathrm{e}^{-mt \Pr(\chi > x)}. \tag{2.6}$$

We note from this expression that the distribution of $X_*(t)$ has an atom at $x = x_0$, with probability mass e^{-mt} (the probability that no jumps occurred before time t).

More generally, exploiting the *Poisson* structure of $N(t)$, for any $n \geqslant 1$, with $x_0 \leqslant x_1 < \cdots < x_n$ and $0 = t_0 \leqslant t_1 < \cdots < t_n$, we get the finite-dimensional distributions

$$\Pr(X_*(t_1) \leqslant x_1, \ldots, X_*(t_n) \leqslant x_n) = \prod_{k=1}^{n} \Pr(X_*(1) \leqslant x_k)^{t_k - t_k - 1} \tag{2.7}$$

with

$$\Pr(X_*(1) \leqslant x) = \mathrm{e}^{-m \Pr(\chi > x)}. \tag{2.8}$$

We now list some properties of the extremal process.

1. *Markov structure*

From (2.5), the process $\{X_*(t), t \geqslant 0\}$ is *Markovian* with global stationary transition probabilities for $t_1 < t_2$

$$\Pr(X_*(t_2) \leqslant x_2 \,|\, X_*(t_1) = x_1) = \mathrm{e}^{-m(t_2-t_1)\Pr(\chi > x_2)} \mathbf{1}(x_1 \leqslant x_2) \tag{2.9}$$

exploiting again the *Poisson* structure of $N(t)$, $t \geqslant 0$. Besides, it is a pure jump process.

More precisely, given that, for some time $s > 0$, $X_*(s) = x_1$, the holding time at x_1 before the next jump is exponentially distributed with frequency parameter $m \Pr(\chi > x_1)$, depending on x_1. Given now that a jump is due to occur at time t under interest, the process $X_*(t)$ jumps from x_1 to $(x_0, x_2]$ with local probability transition

$$K(x_2; x_1) = \left(1 - \frac{\Pr(\chi > x_2)}{\Pr(\chi > x_1)}\right) \cdot \mathbf{1}(x_1 \leqslant x_2) \tag{2.10}$$

independent of time. Indeed, if this is so, and if $X_*(t)$ is *Markov*

$$\partial_t \Pr(X_*(t) \leqslant x_2) = \int K(x_2; x_1) m \Pr(\chi > x_1) \,\mathrm{d}\Pr(X_*(t) \leqslant x_1), \tag{2.11}$$

where the integral is over the variable x_1. From the expression of the transition kernel K, the solution to this *Chapman–Kolmogorov* equation is easily seen to be $\Pr(X_*(t) \leqslant x_2) = \mathrm{e}^{-mt\Pr(\chi > x)}$, as required.

A discrete version of this process may be of interest. Let indeed $(S_n, n \geqslant 1)$ be the times of successive jumps of $X_*(t)$ and let $(X_{*,n} := X_*(S_n), n \geqslant 1)$. The sequence $(X_{*,n}, n \geqslant 1)$ is a discrete *Markov* process with initial distribution $\Pr(X_{*,1} \leqslant x) = \Pr(\chi \leqslant x)$ and transition kernel K.

We next have the recurrence for $n \geqslant 1$,

$$\Pr(X_{*,n+1} \leqslant x_2) = \int K(x_2; x_1) \,\mathrm{d}\Pr(X_{*,n} \leqslant x_1). \tag{2.12}$$

To be complete, some knowledge on the number of jumps, say $N_*(t)$, in the interval $(0, t]$, of the extremal process can next be of some use.

2. *The number of jumps in the extremal process*

It is not hard to see that, with $M_{m-1} := \max(\chi_1, \ldots, \chi_{m-1})$, $M_0 := x_0$,

$$N_*(t) = \sum_{m=1}^{N(t)} \mathbf{1}(\chi_m > M_{m-1}) \tag{2.13}$$

if $N(t) > 0$, zero otherwise. The events

$$(\chi_m > M_{m-1}; m \geqslant 1) := (\chi_m > \chi_1, \ldots, \chi_m > \chi_{m-1}; m \geqslant 1)$$

are called record values for $(\chi_m; m \geqslant 1)$ and are clearly mutually independent. Besides,

$$\Pr(\chi_m > M_{m-1}) = \int \Pr(\chi \leqslant x)^{m-1} \,\mathrm{d}\Pr(\chi \leqslant x) = 1/m. \tag{2.14}$$

Thus $N_*(t)$ is a *Poisson* sum of independent *Bernoulli* random variables $B_m := \mathbf{1}(\chi_m > M_{m-1})$, with $\Pr(B_m = 1) = 1/m$. Given $N(t) = n$, it is well-known and may be checked that

$$\frac{1}{\log n} \sum_{m=1}^{n} B_m \underset{n\uparrow\infty}{\overset{\text{a.s.}}{\to}} 1 \tag{2.15}$$

so that the number of jumps in the extremal process is only of order $\log n$, for large n.

2.2. Inverse of the largest jump process

With $x \geqslant x_0$, let next

$$T(x) := \inf(t > 0 : \Delta(t) > x) \tag{2.16}$$

be the associated time between failure process that is the inverse process of $X_*(t)$.

This process has non-decreasing sample paths (and so is said to be a subordinator, in the sense of *Bochner*). Besides

$$\Pr(T(x) > t) = \Pr(X_*(t) \leqslant x) = e^{-mt \Pr(\chi > x)}. \tag{2.17}$$

Upon exchanging the roles of time and space, we get the inverse process, say $Z(t)$, $t \geqslant t_0 := x_0$, of the largest jump process $X_*(t)$, $t \geqslant 0$.

This process is defined from (2.16) as

$$Z(t) := \inf(s > 0 : X_*(s) > t) \tag{2.18}$$

i.e., as the first time at which $\Delta(\cdot)$ (or $X_*(\cdot)$) exceeds clock time $t \geqslant t_0$.

It is a pure jump process and

$$\Pr(Z(t) > x) = e^{-mx \Pr(\chi > t)}. \tag{2.19}$$

Thus, $Z(t)$ has an exponential distribution with mean value $1/m \Pr(\chi > t)$, and

$$Z(t) \overset{d}{=} 1/(m \Pr(\chi > t)) \cdot E, \tag{2.20}$$

where E is a standard exponential variable such that $\Pr(E > x) = \exp -x$, $x > 0$.

More generally, for any $n \geqslant 1$, with $0 \leqslant x_1 < \cdots < x_n$ and $t_0 := x_0 \leqslant t_1 < \cdots < t_n$,

$$\Pr(Z(t_1) > x_1, \ldots, Z(t_n) > x_n) = \prod_{k=1}^{n} e^{-m(x_k - x_{k-1}) \Pr(\chi > t_k)} \tag{2.21}$$

giving the finite-dimensional distribution of the inverse process.

Under this form, it is clear that $\{Z(t), t \geqslant x_0\}$ is an exponential process with independent (but not stationary) increments. Indeed, from (2.19), we get

$$\mathbf{E} e^{-\lambda Z(t)} = \frac{m \Pr(\chi > t)}{m \Pr(\chi > t) + \lambda}, \tag{2.22}$$

which is the LST of an exponential distribution. This expression admits the alternative ID representation

$$\mathbf{E} e^{-\lambda Z(t)} = \exp - \int_0^{+\infty} (1 - e^{-\lambda x}) \frac{1}{x} e^{-mx \Pr(\chi > t)} \, dx \tag{2.23}$$

using the obvious identity

$$\log(1 + \lambda) = \int_0^{+\infty} (1 - e^{-\lambda x}) \frac{1}{x} e^{-x} \, dx. \tag{2.24}$$

The expression (2.23) turns out to be the *Lévy* representation for an inhomogeneous compound *Poisson* process with independent (but unstationary) increments. For such processes, it is known [46] that, for $t \geqslant t_0$, the following *Lévy-Khintchine* representation holds:

$$\mathbf{E} e^{-\lambda Z(t)} = \mathbf{E} e^{-\lambda Z(t_0)} \cdot \exp - \int_{t_0}^{t} \int_0^{+\infty} (1 - e^{-\lambda x}) \, d\Lambda(s) \otimes d\pi_s(x). \tag{2.25}$$

Here,

$$d\Lambda(t) \otimes d\pi_t(x) \tag{2.26}$$

is the space-time *Lévy* measure for jumps which informs on the rate at which events occur jointly with the jump's height as a function of the occurrence time of this jump.

In (2.25), $\mathbf{E} e^{-\lambda Z(t_0)} = m/(m + \lambda)$ is the LST of an independent initial condition $Z(t_0)$ with exponential distribution of parameter m. Besides, the intensity $\Lambda(t)$, $t \geqslant t_0 := x_0$, is easily identified to be

$$\Lambda(t) = -\log \Pr(\chi > t) \tag{2.27}$$

and the conditional jump's height is exponentially distributed with time-dependent parameter $m \Pr(\chi > t)$, that is

$$d\pi_t(x) = m \Pr(\chi > t) e^{-mx \Pr(\chi > t)} \, dx. \tag{2.28}$$

3. Largest positive jump process of a *Lévy* process and its inverse: the *Fréchet* and inverse *Fréchet* processes

We shall now apply the above findings to a particular class of *Poisson* jump processes, parameterized by $\varepsilon > 0$, which is the lowest endpoint of the jump's height assumed *Pareto* distributed. For a suitable choice of the intensity of the underlying *Poisson* process, in the limit $\varepsilon \downarrow 0$, we shall find the self-similar *Lévy–Fréchet* processes for the associated sum and extremal processes.

3.1. One-sided Lévy-stable processes

Assume $X(t)$ is a standard one-sided Lévy-stable process [3]. Then, with $\alpha \in (0,1)$, Γ the *Euler* function and $a := \Gamma(1-\alpha) > 0$,

$$\mathbf{E}\,\mathrm{e}^{-\lambda X(t)} = \mathrm{e}^{-at\lambda^{\alpha}}. \tag{3.1}$$

Note from (3.1) that

$$X(t) \overset{d}{=} t^{1/\alpha} L_{\alpha}, \quad t \geqslant 0, \tag{3.2}$$

where L_{α} is a standard positive *Lévy* variable, with

$$\mathbf{E}\,\mathrm{e}^{-\lambda L_{\alpha}} = \exp -a\lambda^{\alpha}. \tag{3.3}$$

This process is known to be with stationary independent increments and self-similar with scaling exponent $1/\alpha$. Besides, its sample paths are non-decreasing and is therefore a subordinator. However, it is not, strictly speaking, in the compound *Poisson* class. Rather, a *Lévy* process is a limiting compound *Poisson* process [8,46] and we shall now recall in which sense.

First recall the identity (which may be checked upon integrating by part)

$$\int_0^{+\infty} (1-\mathrm{e}^{-\lambda x})\,\mathrm{d}\pi(x) = a\lambda^{\alpha}, \quad \text{with } \pi(x) = -x^{-\alpha},\ x > 0. \tag{3.4}$$

Here $\mathrm{d}\pi(x)$ is a positive *Radon* exponent measure on $(0,+\infty)$ (known as *Lévy* spectral measure), with infinite total mass, due to the algebraic divergence of its density in the vicinity of zero. As this density: $\mathrm{d}\pi(x)/\mathrm{d}x = \alpha x^{-(1+\alpha)}$ is not a probability density, a *Lévy* process, for which (3.1) holds, is not stricto sensu a compound *Poisson* process (compare with (2.1)); rather, it can be obtained from a "coarse" compound *Poisson* process $X^{\varepsilon}(t)$ in the limit $\varepsilon \downarrow 0$.

Let indeed $\varepsilon > 0$; consider the compound *Poisson* process $X^{\varepsilon}(t)$, $t \geqslant 0$, defined by an exponentially distributed holding time with parameter m_{ε} and with iid positive increments, say χ^{ε}, with normalized truncated probability distribution

$$\Pr(\chi^{\varepsilon} > x) = \frac{x^{-\alpha}}{m_{\varepsilon}} \cdot \mathbf{1}(x \geqslant \varepsilon). \tag{3.5}$$

We note that this is a *Pareto* distribution in the heavy-tailed class with tail index α: jumps with very large amplitude are very likely to occur in the sample paths of $X^{\varepsilon}(t)$.

From (3.5) the normalization constant is easily obtained. It is

$$m_{\varepsilon} = \int_{\varepsilon}^{+\infty} \alpha x^{-(1+\alpha)}\,\mathrm{d}x = \varepsilon^{-\alpha} \tag{3.6}$$

and m_{ε} tends to infinity as ε tends to zero. Next, we form the quantity $\mathbf{E}\,\mathrm{e}^{-\lambda X^{\varepsilon}(t)}$. From (2.1), it is, with $\mathbf{E}\,\mathrm{e}^{-\lambda\chi^{\varepsilon}}$, the *Laplace* transform of χ^{ε}'s distribution

$$\mathbf{E}\,\mathrm{e}^{-\lambda X^{\varepsilon}(t)} = \mathrm{e}^{-m_{\varepsilon}t(1-\mathbf{E}\,\mathrm{e}^{\lambda\chi^{\varepsilon}})}. \tag{3.7}$$

Now, from (3.5) and (3.6)

$$m_{\varepsilon}(1-\mathbf{E}\,\mathrm{e}^{\lambda\chi^{\varepsilon}}) = \int_{\varepsilon}^{+\infty} (1-\mathrm{e}^{-\lambda x})\,\mathrm{d}\pi(x) \underset{\varepsilon\downarrow 0}{\rightarrow} a\lambda^{\alpha}, \tag{3.8}$$

which is consistent with (3.1) and (3.4).

Thus, $X(t)$ defined from (3.1) is the limiting compound *Poisson* process $X^{\varepsilon}(t)$ as $\varepsilon \downarrow 0$ and is indeed a process with iid increments.

Compound *Poisson* subordinators exhibit finitely many isolated jumps on finite time intervals. This is not the case for *Lévy* ones: the many jumps with tiny amplitudes contribute in the limit to a *Hölder* continuous drift, with *Hölder* exponents in the range $[0, 1/\alpha]$ [17], as a result of jumps' clustering in the limit. This drift is punctuated with a few very large *Pareto*-like jumps. Globally, from (3.2), the process drifts to infinity much faster than clock time t, as a result of the very large jumps which occur.

We shall now apply the constructions of Section 2 to the coarse compound *Poisson* version $X^{\varepsilon}(t)$ of the *Lévy* subordinator $X(t)$. As $\varepsilon \downarrow 0$, we shall get a limiting version of the associated extremal process, with its inverse. We shall find, in the limit, processes to be identified with *Fréchet* and inverse *Fréchet* processes [39,40]. Some of their remarkable properties are then addressed.

3.2. Fréchet process and inverse in the range $\alpha \in (0,1)$

Consider the coarse compound *Poisson* version $X^{\varepsilon}(t)$ of the *Lévy* subordinator $X(t)$ defined previously with index $\alpha \in (0,1)$. With $m_{\varepsilon} = \varepsilon^{-\alpha}$ defined in (3.6), and jumps' height defined in (3.5), with range $x \geqslant \varepsilon := x_0$.

With $N^{\varepsilon}(t)$ a *Poisson* process with intensity $m_{\varepsilon}t$, define the associated jump process by

$$\Delta^{\varepsilon}(t) := \varepsilon \cdot \mathbf{1}(N^{\varepsilon}(t) = 0) + [\chi^{\varepsilon}_{N^{\varepsilon}(t)}] \cdot \mathbf{1}(N^{\varepsilon}(t) > 0), \quad t \geqslant 0. \tag{3.9}$$

Following Section 2, the associated extremal process is therefore

$$X^{\varepsilon}_{*}(t) = \max(\varepsilon, \chi^{\varepsilon}_1, \ldots, \chi^{\varepsilon}_{N^{\varepsilon}(t)}). \tag{3.10}$$

For any $n \geqslant 1$, its finite-dimensional distribution is

$$\Pr(X^{\varepsilon}_{*}(t_1) \leqslant x_1, \ldots, X^{\varepsilon}_{*}(t_n) \leqslant x_n) = \prod_{k=1}^{n} \mathrm{e}^{-(t_k - t_{k-1})x_k^{-\alpha}} \tag{3.11}$$

with $\varepsilon = x_0 \leqslant x_1 < \cdots < x_n$, $0 = t_0 \leqslant t_1 < \cdots < t_n$.

As a result, with $x \geqslant \varepsilon$, we get

$$\Pr(T^{\varepsilon}(x) > t) = \Pr(X_{*}(t) \leqslant x) = \mathrm{e}^{-m_{\varepsilon}t \Pr(\chi^{\varepsilon} > x)} = \mathrm{e}^{-tx^{-\alpha}}, \tag{3.12}$$

where

$$T^{\varepsilon}(x) := \inf(t > 0 : \Delta^{\varepsilon}(t) > x) \tag{3.13}$$

is the time to failure process of $\Delta^{\varepsilon}(t)$. Upon exchanging the roles of time and space, we get the inverse process, say $Z^{\varepsilon}(t)$, $t \geqslant \varepsilon$, of the largest positive jump process $\Delta^{\varepsilon}(t)$, $t \geqslant 0$, and

$$\Pr(Z^{\varepsilon}(t) > x) = \mathrm{e}^{-xt^{-\alpha}}. \tag{3.14}$$

More generally, with $0 \leqslant x_1 < \cdots < x_n$ and $\varepsilon = t_0 \leqslant t_1 < \cdots < t_n$

$$\Pr(Z^{\varepsilon}(t_1) > x_1, \ldots, Z^{\varepsilon}(t_n) > x_n) = \prod_{k=1}^{n} \mathrm{e}^{-(x_k - x_{k-1})t_k^{-\alpha}} \tag{3.15}$$

gives the finite-dimensional distributions of the inverse process. As $\varepsilon \downarrow 0$, we get

$$\{X^{\varepsilon}_{*}(t), t \geqslant 0\} \xrightarrow{d} \{X_{*}(t), t \geqslant 0\}, \quad \{Z^{\varepsilon}(t), t \geqslant 0\} \xrightarrow{d} \{Z(t), t \geqslant 0\} \tag{3.16}$$

with the limiting process $\{X_{*}(t), t \geqslant 0\}$ and $\{Z(t), t \geqslant 0\}$ defined by their finite-dimensional distributions. Sometimes we shall index the limiting extremal process by α, say according to $X_{*}(t) = X_{*,\alpha}(t)$, to underline its dependence on the parameter α.

More precisely, for any $n \geqslant 1$, and $0 \leqslant x_1 < \cdots < x_n$, $0 \leqslant t_1 < \cdots < t_n$, these finite-dimensional distributions are given by

$$\Pr(X_{*,\alpha}(t_1) \leqslant x_1, \ldots, X_{*,\alpha}(t_n) \leqslant x_n) = \prod_{k=1}^{n} \mathrm{e}^{-(t_k - t_{k-1})x_k^{-\alpha}} \tag{3.17}$$

and

$$\Pr(Z(t_1) > x_1, \ldots, Z(t_n) > x_n) = \prod_{k=1}^{n} \mathrm{e}^{-(x_k - x_{k-1})t_k^{-\alpha}} \tag{3.18}$$

with the convention $t_0 = x_0 = 0$.

We note in particular that

$$\Pr(X_*(t) \leqslant x) = \Pr(X_*(1) \leqslant x)^t \tag{3.19}$$

with

$$\Pr(X_*(1) \leqslant x) = \mathrm{e}^{-x^{-\alpha}}, \quad x > 0, \tag{3.20}$$

a *Fréchet* distribution [6,9].

For this reason, the limiting extremal process $\{X_*(t), t \geqslant 0\}$ is called a *Fréchet* process and we have found that, when $\alpha \in (0,1)$, the extremal *Fréchet* process is the largest positive jump process associated to a *Lévy* α-stable subordinator. Note from (3.19) and (3.20) that

$$X_*(t) \stackrel{d}{=} t^{1/\alpha} X_*(1), \quad t \geqslant 0. \tag{3.21}$$

This process drifts to infinity faster than clock time t.

3.2.1. Additional properties of the Fréchet process

We list below some statistical properties of interest (see also [6,39]).

1. *Markov structure*

From (3.19) and (3.20), the distribution of $X_*(t)$ has no atom at $x = 0$. In the vicinity of $t = 0$ the sample paths of $X_*(t)$ are *Hölder* continuous: only after some finite time, say $\tau_0 = 1$, is $(X_*(t), t \geqslant \tau_0)$ a pure jump *Markov process*. This process is stochastically continuous with right-continuous sample paths [39].

More precisely, from the previous construction, given that for some time $s > 0$, $X_*(s) = x_1$, the holding time at x_1 before the next jump is exponentially distributed with frequency parameter $x_1^{-\alpha}$, depending on x_1. Given now that a jump is due to occur at time t under interest, the process $X_*(t)$ jumps from x_1 to $(0, x_2]$ with local probability transition

$$K(x_2; x_1) = \left(1 - \left(\frac{x_2}{x_1}\right)^{-\alpha}\right) \cdot \mathbf{1}(x_1 \leqslant x_2) \tag{3.22}$$

independent of time.

2. *Multiplicative structure*

A discrete version of this process may be of interest. Let indeed $(S_n, n \geqslant 1)$ be the times of successive jumps of $X_*(t)$ after time $\tau_0 = 1$, and let

$$(X_{*,n} := X_*(S_n), n \geqslant 1), \quad X_{*,0} := X_*(\tau_0 = 1). \tag{3.23}$$

The sequence $(X_{*,n}, n \geqslant 0)$ is a discrete *Markov* process with initial distribution $\Pr(X_{*,0} \leqslant x) = \exp -x^{-\alpha}$ and instantaneous transition kernel K.

We therefore have the recurrence for $n \geqslant 0$

$$\Pr(X_{*,n+1} \leqslant x_2) = \int_0^{x_2} \left(1 - \left(\frac{x_2}{x_1}\right)^{-\alpha}\right) \mathrm{d}\,\Pr(X_{*,n} \leqslant x_1). \tag{3.24}$$

Observing that

$$\left(1 - \left(\frac{x_2}{x_1}\right)^{-\alpha}\right) = \Pr\left(P \leqslant \frac{x_2}{x_1}\right),$$

where P is a *Pareto* random variable with support $(1, \infty)$, with exponent α, we get, upon iterating, that $X_{*,n}$ has the multiplicative structure

$$X_{*,n} \stackrel{d}{=} X_{*,0} \prod_{m=1}^{n} P_m, \tag{3.25}$$

where $(P; P_m, m \geqslant 1)$ is an iid sequence of *Pareto* variables such that $\Pr(P > x) = x^{-\alpha}$, $x \geqslant 1$, independent of the initial *Fréchet* variable $X_{*,0}$.

There remains to say a few words on the number of jumps $N_*(1,t]$, in the interval $(1,t]$, of the extremal process $X_*(t)$. Following [39], it is non-homogeneous *Poisson* process with intensity $\mathbf{E}N_*(1,t] = \log t$, in such a way that

$$\Pr(N_*(1,t] = n) = \frac{1}{n!\cdot t}(\log t)^n, \quad n \geqslant 0. \tag{3.26}$$

As time goes by, the occurrence of a new record value becomes less probable.

Finally we have the following multiplicative description of $X_*(t)$:

$$X_*(t) \stackrel{d}{=} X_*(1) \prod_{m=1}^{N_*(1,t]} P_m, \tag{3.27}$$

which is the multiplicative structure property.

3. *Self-similarity*

The *Fréchet* extremal process $X_*(t)$ is self-similar in the sense that, for any $n \geqslant 1$, $\gamma > 0$, and $0 \leqslant t_1 < \cdots < t_n$, we have

$$(X_{*,\alpha}(\gamma t_1), \ldots, X_{*,\alpha}(\gamma t_n)) \stackrel{d}{=} \left(\gamma^{1/\alpha} X_{*,\alpha}(t_1), \ldots, \gamma^{1/\alpha} X_{*,\alpha}(t_n)\right), \tag{3.28}$$

which may be checked from (3.17). The self-similarity (scaling) exponent is $1/\alpha$. The *Fréchet* extremal process is thus an example of a self-similar *Markov* process.

3.2.2. Additional properties of the inverse Fréchet process

We list below some statistical properties of interest (see also [40]).

1. *Independent unstationary increments*

From the previous study, the inverse *Fréchet* process $\{Z(t), t \geqslant 0\}$ has independent increments with space-time *Lévy* measure for jumps $\mathrm{d}\Lambda(t) \otimes \mathrm{d}\pi_t(x)$, in the sense that

$$\mathbf{E}\,\mathrm{e}^{-\lambda Z(t)} = \frac{1}{1+\lambda t^\alpha} = \exp - \int_0^t \int_0^{+\infty} (1 - \mathrm{e}^{-\lambda x})\, \mathrm{d}\Lambda(s) \otimes \mathrm{d}\pi_s(x). \tag{3.29}$$

The intensity $\Lambda(t, t+\tau) := \int_t^{t+\tau} \mathrm{d}\Lambda(s)$, for $t > 0$, $\tau > 0$, is easily identified to be

$$\Lambda(t, t+\tau) = -\log \frac{(t+\tau)^{-\alpha}}{t^{-\alpha}} = \alpha \log \frac{t+\tau}{t}. \tag{3.30}$$

As $t \downarrow 0$, the expected number of jumps $\Lambda(t, t+\tau)$ in the interval $(t, t+\tau]$ tends to infinity like $-\log(t/\tau)$: as for the extremal process, in an inverse *Fréchet* process, the expected number of jumps tends to infinity in the vicinity of $t = 0$.

The conditional jump' height is exponentially distributed with time-dependent mean value t^α

$$\mathrm{d}\pi_t(x) = t^{-\alpha} \mathrm{e}^{-xt^{-\alpha}}\, \mathrm{d}x. \tag{3.31}$$

Globally, with E an exponential variable, $\{Z(t), t \geqslant 0\}$ is such that

$$Z(t) \stackrel{d}{=} t^\alpha \cdot E, \quad t \geqslant 0, \tag{3.32}$$

as

$$\Pr(Z(t) > x) = \mathrm{e}^{-xt^{-\alpha}}, \quad x, t > 0. \tag{3.33}$$

As time goes to infinity, it drifts much slower than clock time.

2. *Self-similarity*

The inverse *Fréchet* process $Z(t)$ is also self-similar in the sense that, for any $n \geqslant 1$, $\gamma > 0$, and $0 \leqslant t_1 < \cdots < t_n$, we have

$$(Z(\gamma t_1), \ldots, Z(\gamma t_n)) \stackrel{d}{=} (\gamma^\alpha Z(t_1), \ldots, \gamma^\alpha Z(t_n)), \tag{3.34}$$

which may be checked from (3.18). The self-similarity (scaling) exponent is α. The inverse *Fréchet* process is thus an example of a self-similar process with independent unstationary increments.

3.2.3. A connection between the direct and inverse Fréchet processes

An additional property which may be checked from (3.17) and (3.18), connecting the extremal *Fréchet* process $X_*(t)$ and its inverse $Z(t)$, is the following: the process

$$(Z(t^{-1/\alpha})^{-1/\alpha},\ t > 0) \tag{3.35}$$

is a left-continuous version of the *Fréchet* extremal process. In particular, from (3.33)

$$\Pr(Z(t^{-1/\alpha})^{-1/\alpha} \leqslant x) = \mathrm{e}^{-tx^{-\alpha}}. \tag{3.36}$$

Thus the extremal *Fréchet* process is some "output" of an inverse *Fréchet* process after some suitable deterministic change of time.

3.3. Fréchet process and inverse in the range α > 1

Fréchet processes are also defined in the parameter range $\alpha > 1$. Let us now exhibit the connection between such processes and *Lévy*-stable subordinators.

If $\alpha > 1$, we start with the *Lévy* subordinator with index $1/\alpha < 1$. Following the preceding steps, we construct the associated extremal *Fréchet* process, say $X_{*,1/\alpha}(t)$ with finite-dimensional distribution

$$\Pr(X_{*,1/\alpha}(t_1) \leqslant y_1, \ldots, X_{*,1/\alpha}(t_n) \leqslant y_n) \prod_{k=1}^{n} \mathrm{e}^{-(t_k - t_{k-1}) y_k^{-1/\alpha}}. \tag{3.37}$$

Then, consider the *output* extremal process

$$\{X_{*,\alpha}(t) := X_{*,1/\alpha}(t)^{1/\alpha^2}, t \geqslant 0\}. \tag{3.38}$$

We get

$$\Pr(X_{*,\alpha}(t_1) \leqslant x_1, \ldots, X_{*,\alpha}(t_n) \leqslant x_n) = \prod_{k=1}^{n} \mathrm{e}^{-(t_k - t_{k-1}) x_k^{-\alpha}}, \tag{3.39}$$

which is a *Fréchet* process with parameter $\alpha > 1$.

Thus, a *Fréchet* process with parameter $\alpha > 1$ is the largest positive jump process associated to a *Lévy* subordinator with index $1/\alpha < 1$, raised to the power $1/\alpha^2$. The inverse *Fréchet* process in this parameter range is similar to the one discussed previously in the range $\alpha \in (0, 1)$. However, as $\alpha > 1$, it drifts to infinity much faster than clock time.

Remark 1. As is well-known, the *Lévy* subordinator $X(t)$ defined in (3.1) is ill-defined when $\alpha \geqslant 1$ (this may be checked while observing that in this case the LST is not a completely monotone function of the Laplace argument). However, the coarse compound Poisson version $X^{\varepsilon}(t)$ defined in (3.9) still makes sense, even if $\alpha \geqslant 1$. The associated jump and extremal processes $\Delta^{\varepsilon}(t)$ and $X_*^{\varepsilon}(t)$ are also well-defined and $X_*(t) := \lim X_*^{\varepsilon}(t)$ does exist in the limit $\varepsilon \downarrow 0$, even for $\alpha \geqslant 1$. The latter is obviously the *Fréchet* process for $\alpha \geqslant 1$, with finite-dimensional distributions given by (3.39). This stands for an alternative construction of *Fréchet* and inverse *Fréchet* processes in the range $\alpha \geqslant 1$.

3.4. Related extremal process: Gumbel process

It is a very common feature in Physics that *observables* are modeled through the logarithms of other quantities with much more "appealing" meaning, such as the "energy" or magnitude of some underlying phenomenon or process. The distinctive feature of the logarithmic scale for observable is that it measures the distance between two magnitudes through their ratio rather than their difference; this transformation thus supplies a discrimination power between two physical signals which is insensitive to their absolute intensities in that it essentially deals with their ratio. Consequently, it is argued that the logarithm of the *Fréchet* extremal process should stand for most relevant statistical models for the extremal magnitude process measured in logarithmic scale. We shall now say a few words on this process.

If $\alpha > 0$, $\alpha \neq 1$, consider the output process

$$Y_*(t) = \alpha \log X_{*,\alpha}(t). \tag{3.40}$$

We note from (3.38) that if $\alpha > 1$, this process is also $Y_*(t) = 1/\alpha \log X_{*,1/\alpha}(t)$, with $X_{*,1/\alpha}(t)$ the largest positive jump process associated to a *Lévy* subordinator with index $1/\alpha < 1$.

From the definition of $Y_*(t)$, its finite-dimensional distributions may be computed from the ones of the *Fréchet* extremal process. In particular,

$$\Pr(Y_*(t) \leqslant x) = \mathrm{e}^{-t\mathrm{e}^{-x}} \tag{3.41}$$

with $\Pr(Y_*(1) \leqslant x) = \mathrm{e}^{-\mathrm{e}^{-x}}$, $x \in (-\infty, +\infty)$, a *Gumbel* distribution [12]. Thus $Y_*(t)$ is identified with the *Gumbel* extremal process. This process is *Markov* with local transition kernel $K(x_2; x_1) = (1 - \mathrm{e}^{-(x_2 - x_1)} \cdot \mathbf{1}(x_1 \leqslant x_2)$, exponential holding time at x_1, with parameter e^{-x_1}. It has the additive structure

$$Y_* \overset{d}{=} Y_*(1) + \sum_{m=1}^{N_*(1;t]} E_m \tag{3.42}$$

with $(E; E_m, m \geqslant 1)$ a sequence of iid exponentially distributed exponential random variables, independent of the initial *Gumbel* variable $Y_*(1)$. Besides, from (3.41) we have

$$Y_*(t) \overset{d}{=} \log t + Y_*(1), \quad t > 0. \tag{3.43}$$

This process drifts to infinity like the logarithm of clock time.

The *Gumbel* extremal process is also self-similar, but in the broad sense that is, it satisfies: for any $n \geqslant 1$, $\gamma > 0$, and $0 \leqslant t_1 < \cdots < t_n$,

$$(Y_*(\gamma t_1), \ldots, Y_*(\gamma t_n)) \overset{d}{=} (Y_*(t_1) + \log \gamma, \ldots, Y_*(t_n) + \log \gamma), \tag{3.44}$$

which may be checked from (3.41). This self-similarity is translational (or broad sense) and not of the scaling type.

4. *Lévy* and *Fréchet* Geometric stable processes

In this section, we introduce a related class of self-similar processes, namely the class of Geometric *Lévy*-stable and Geometric *Fréchet* max-stable processes. They can be obtained while "subordinating" the previous *Lévy*- and *Fréchet*-stable processes in suitable random Exponential time (see [5,31,37,38] for motivations on subordination in Economy and Finance and also [14] for some related use in Physics).

4.1. A random Exponential model of time

Define an Exponential process, say $\{E(t), t \geqslant 0\}$ as a particular inverse *Fréchet* process, setting $\alpha = 1$ in (3.18). From (3.29),

$$\mathbf{E}\,\mathrm{e}^{-\lambda E(t)} = 1/(1 + \lambda t).$$

This process has independent unstationary increments with space-time *Lévy* measure for jumps $\mathrm{d}\Lambda(t) \otimes \mathrm{d}\pi_t(x)$, in the sense that

$$\mathbf{E}\,\mathrm{e}^{-\lambda E(t)} = \exp - \int_0^t \int_0^{+\infty} (1 - \mathrm{e}^{-\lambda x})\, \mathrm{d}\Lambda(s) \otimes \mathrm{d}\pi_s(x). \tag{4.1}$$

The intensity $\Lambda(t, t+\tau) := \int_t^{t+\tau} \mathrm{d}\Lambda(s)$, for $t > 0$, $\tau > 0$, is now identified to be

$$\Lambda(t, t+\tau) = -\log \frac{(t+\tau)^{-1}}{t^{-1}} = \log \frac{t+\tau}{t}. \tag{4.2}$$

The conditional jump distribution is exponentially distributed with time-dependent mean value t

$$\mathrm{d}\pi_t(x) = t^{-1} \mathrm{e}^{-xt^{-1}} \mathrm{d}x. \tag{4.3}$$

In this model, the expected number of jumps in the interval $(1, 1+\tau]$ is $\Lambda(1, 1+\tau) = \log(1+\tau) \sim \log \tau$ for large τ: the moves of $\{E(t), t \geqslant 0\}$ get sparser and sparser as time goes by; however, the larger clock time is, the larger the amplitude of the moves is.

Globally, with E an exponential variable, $\{E(t), t \geqslant 0\}$ is such that

$$E(t) \overset{d}{=} t \cdot E, \quad t > 0, \tag{4.4}$$

as a result of

$$\Pr(E(t) > x) = e^{-xt^{-1}}, \quad x, t > 0. \tag{4.5}$$

Globally, this process therefore closely looks like clock time itself. Besides, this Exponential process $E(t)$ is also self-similar (with scaling parameter $\alpha = 1$) in the sense that, for any $n \geqslant 1$, $\gamma > 0$, and $0 \leqslant t_1 < \cdots < t_n$, we have

$$(E(\gamma t_1), \ldots, E(\gamma t_n)) \stackrel{d}{=} (\gamma E(t_1), \ldots, \gamma E(t_n)). \tag{4.6}$$

From all these properties, it turns out that the Exponential process is a good model of time as a random process.

4.2. Self-similar processes in random exponential time: subordination

4.2.1. The Geometric Lévy-stable process

Let $\{X(t), t \geqslant 0\}$ be, as in (3.1), a *Lévy*-stable process with exponent $\alpha \in (0, 1)$. Assume it is independent of the Exponential process just defined $\{E(t), t > 0\}$.

Consider next the "subordinated" process $\{G(t) := X(E(t)), t > 0\}$. This process has independent unstationary increments. Besides, as it can easily be checked

$$\mathbf{E}\, e^{-\lambda G(t)} = \frac{1}{1 + at\lambda^{\alpha}}. \tag{4.7}$$

As a composition of two self-similar processes, this process is itself self-similar. Its scaling exponent is $1/\alpha$, in the sense that, for any $\gamma > 0$

$$(G(\gamma t_1), \ldots, G(\gamma t_n)) \stackrel{d}{=} \left(\gamma^{1/\alpha} G(t_1), \ldots, \gamma^{1/\alpha} G(t_n)\right). \tag{4.8}$$

The process $\{G(t), t \geqslant 0\}$ is known as the geometric *Lévy*-stable process. The random variable with LST: $1/(1 + a\lambda^{\alpha})$ is known in the literature as a geometric stable random variable (see for example [18,19] and the references therein).

Additional insight into the process $G(t)$ now arises from the following findings.

With $x \geqslant 0$, $\alpha \in (0, 1)$, following [8], let

$$\phi_{\alpha}(x) := \sum_{n \geqslant 0} \frac{1}{\Gamma(1 + n\alpha)} (-x)^n \tag{4.9}$$

be the (entire) *Mittag–Leffler* function (which reduces to the exponential function if $\alpha \uparrow 1$).

Define a positive *Mittag–Leffler* random variable, say M, through

$$\Pr(M > x) = \phi_{\alpha}(x^{\alpha}), \tag{4.10}$$

which is consistent with the fact that the *Mittag–Leffler* function is decreasing.

The *Laplace* transform $\mathbf{E}\, e^{-\lambda M}$ of such *Mittag–Leffler* variables is given by the formula [8,35]

$$\mathbf{E}\, e^{-\lambda M} = \frac{1}{1 + \lambda^{\alpha}}. \tag{4.11}$$

From (4.7) we readily get

$$G(t) \stackrel{d}{=} (at)^{1/\alpha} M, \quad t \geqslant 0, \tag{4.12}$$

so that $\{G(t), t \geqslant 0\}$ is a *Mittag–Leffler* process as well.

It may now be checked, upon integrating by parts, that the *Laplace–Stieltjes* transform of the function $\phi_{\alpha}(x^{\alpha})$ is given by

$$\int_0^{+\infty} e^{-\lambda x} \phi_{\alpha}(x^{\alpha})\, dx = \frac{1}{\lambda(1 + \lambda^{-\alpha})}. \tag{4.13}$$

As a corollary of this identity, we get, as an extension of (2.24):

$$\int_0^{+\infty} (1 - e^{-\lambda x}) \frac{\alpha}{x} \phi_{\alpha}(x^{\alpha})\, dx = \log(1 + \lambda^{\alpha}). \tag{4.14}$$

Now, from (4.7) and after some scaling transform

$$\mathbf{E}\,\mathrm{e}^{-\lambda G(t)} = \exp - \log(1 + at\lambda^{\alpha}) = \exp - \int_0^{+\infty} -(1 - \mathrm{e}^{-\lambda x}) \frac{\alpha}{x} \phi_{\alpha}\left(\frac{x^{\alpha}}{at}\right) \mathrm{d}x. \tag{4.15}$$

This expression allows us to identify the space-time *Lévy* measure for jumps $\mathrm{d}\Lambda(t) \otimes \mathrm{d}\pi_t(x)$ of the process $G(t)$, for which the *Lévy–Khintchine* representation holds

$$\mathbf{E}\,\mathrm{e}^{-\lambda G(t)} = \frac{1}{1 + at\lambda^{\alpha}} = \exp - \int_0^t \int_0^{+\infty} (1 - \mathrm{e}^{-\lambda x})\, \mathrm{d}\Lambda(s) \otimes \mathrm{d}\pi_s(x). \tag{4.16}$$

We find, after some easy computations

$$\Lambda(t) = \log t. \tag{4.17}$$

As $t \downarrow 0$, the expected number of jumps $\Lambda(t)$ in the interval $(0, t]$ also tends to infinity: as for the inverse *Fréchet* process, the expected number of jumps tends to infinity in the vicinity of $t = 0$.

The conditional jumps' height has the time-dependent *Mittag–Leffler* distribution

$$\int_x^{\infty} \mathrm{d}\pi_t(z) = \phi_{\alpha}\left(\frac{x^{\alpha}}{at}\right). \tag{4.18}$$

From the behavior $\phi_{\alpha}(x^{\alpha}) \sim x^{-\alpha}$ for large x, we get that for large times, the median value of the jumps' height now grows like $t^{1/\alpha}$ (note that the mean itself does not take finite values).

4.2.2. The Geometric Fréchet process

Let now $\{X_*(t), t > 0\}$ be a *Fréchet*-stable process with exponent $\alpha > 0$, independent of $\{E(t), t > 0\}$.

Consider next the subordinated process $\{G_*(t) := X_*(E(t)), t > 0\}$. Clearly, this process is *Markovian* with transition kernel and rate easily obtainable. We note in particular that

$$\Pr(G_*(t) \leqslant x) = \int_0^{\infty} \mathrm{e}^{-sx^{-\alpha}} \frac{1}{t} \mathrm{e}^{-st^{-1}}\, \mathrm{d}s = \frac{1}{1 + tx^{-\alpha}} \tag{4.19}$$

so that

$$G_*(t) \stackrel{d}{=} t^{1/\alpha} G_*(1) \tag{4.20}$$

with $G_*(1)$ a random variable such that $\Pr(G_*(1) \leqslant x) = (1 + x^{-\alpha})^{-1}$.

As a composition of self-similar processes, $G_*(t)$ is also self-similar with exponent $1/\alpha$, in the sense that

$$(G_*(\gamma t_1), \ldots, G_*(\gamma t_n)) \stackrel{d}{=} \left(\gamma^{1/\alpha} G_*(t_1), \ldots, \gamma^{1/\alpha} G_*(t_n)\right). \tag{4.21}$$

The process $\{G_*(t), t \geqslant 0\}$ is known as the geometric-*Fréchet*-stable process, as the random variable with probability distribution function $1/(1 + x^{-\alpha})$ is known in the literature as a geometric (max-) stable random variable (see for example [32,36] and the references therein).

4.2.3. The Geometric Gumbel (or logistic) process

Consider finally the output process

$$L_*(t) = \alpha \log G_*(t). \tag{4.22}$$

Its finite-dimensional distributions may be computed from the ones of the geometric *Fréchet* process: In particular, from (4.19) and (4.22)

$$\Pr(L_*(t) \leqslant x) = \frac{1}{1 + t\mathrm{e}^{-x}}. \tag{4.23}$$

We note that $\Pr(L_*(1) \leqslant x) = (1 + \mathrm{e}^{-x})^{-1}$, $x \in (-\infty, +\infty)$. This probability distribution function is the one of a logistic distribution. Thus $L_*(t)$ is identified with the logistic extremal process. Just like *Gumbel* processes, it is a self-similar process of the translational type in the sense that, for any $\gamma > 0$,

$$(Y_*(\gamma t_1), \ldots, Y_*(\gamma t_n)) \stackrel{d}{=} (Y_*(t_1) + \log\gamma, \ldots, Y_*(t_n) + \log\gamma). \tag{4.24}$$

Remark 2. Exploiting the connection (3.35) between extremal processes and their inverses, defining as usual the inverse extremal geometric process (i.e., the inverse process of $G_*(t)$) to be $P(t) := \inf(s > 0 : G_*(s) > t)$, we get

$$\Pr(P(t) > x) = \frac{1}{1 + xt^{-\alpha}}. \tag{4.25}$$

This process has unstationary independent increments and

$$P(t) \stackrel{d}{=} t^{\alpha} P \tag{4.26}$$

with P a generalized Pareto random variable (with exponent one), i.e., with probability distribution $\Pr(P > x) = (1 + x)^{-1}$.

Remark 3 (*Related stationary processes*). So far, we have been chiefly concerned with self-similar motions, either *Fréchet* or *Lévy*. It is interesting to consider such motions in the linear force (*Langevin*) context. We shall formulate this problem using the so-called *Lamperti* transform, inspired from group theory. This approach emphasizes the fact that self-similarity and stationarity are closely related: an exponential time-transform translates scale-invariance into shift-invariance of the stationary process.

As is well-known from [21], if the process $Z_H(t)$, $t \geqslant 0$, is self-similar with exponent $H > 0$, the process

$$U_H(t) := \mathrm{e}^{-tH} \cdot Z_H(\mathrm{e}^t) \tag{4.27}$$

is stationary with the distribution of $Z_H(1)$ as invariant probability distribution under shift in time.

The *Lévy*, *Fréchet*, inverse *Fréchet* together with their geometrical extensions are all self-similar with scaling exponent H either α or $1/\alpha$. Performing the *Lamperti* transform, we are left with a bunch of stationary processes with invariant probability of the *Lévy*, *Fréchet*, exponential or geometric stable distributions.

5. *Fréchet* and *Lévy* semi-stable processes

We shall now extend the above constructions which rest upon a weaker notion of stability, which is "semi-stability". A construction of semi-stable *Fréchet* and *Lévy* processes is then supplied.

5.1. *Fréchet and inverse Fréchet semi-stable processes*

Let $c, \varepsilon > 0$. Let v be a bounded periodic function with period $-\log c$ on the real line. Suppose in addition that function v is such that $z - v(z)$ is non-decreasing.

Consider now a random variable χ^{ε} with probability distribution

$$\Pr(\chi^{\varepsilon} > x) = \frac{1}{m_{\varepsilon}} x^{-\alpha} \exp \alpha v(\log x), \quad x \geqslant \varepsilon = x_0. \tag{5.1}$$

Note that the required conditions on v are consistent with the fact this expression is indeed a probability distribution (see the remark below for additional justifications). In this model for jumps' height, a plot of $-\log \Pr(\chi^{\varepsilon} > x)$ against $\log x$ should exhibit periodic oscillations around a linear trend with positive slope α.

In (5.1), m_{ε} is the normalizing constant. Remark also that function $v = 0$ satisfies all the required conditions so that in this "trivial" case we are left with the previous *Pareto* model (3.5) for χ^{ε}.

Extending (3.4), define the *Lévy* spectral function associated to (5.1) as

$$\pi(x) = -x^{-\alpha} \mathrm{e}^{\alpha v(\log x)}, \quad x \in (0, +\infty). \tag{5.2}$$

It may be checked that such $\pi(x)$ are those which satisfy the scaling condition

$$\sigma^{-n} \pi(x) = \pi(\sigma^{n/\alpha} x) \quad \text{for all } n \in \mathbb{Z} \tag{5.3}$$

with $\sigma := c^{-\alpha}$.

With $N^{\varepsilon}(t)$ a *Poisson* process with intensity $m_{\varepsilon} t$, define now a new jump process as in (3.9) by

$$\Delta^{\varepsilon}(t) = \varepsilon \cdot \mathbf{1}(N^{\varepsilon}(t) = 0) + [\chi^{\varepsilon}_{N^{\varepsilon}(t)}] \cdot \mathbf{1}(N^{\varepsilon}(t) > 0), \quad t \geqslant 0. \tag{5.4}$$

Proceeding as before, with $\{X_*^{\varepsilon}, t \geqslant 0\}$ the associated extremal process, we get

$$\{X_*^{\varepsilon}(t), t \geqslant 0\} \xrightarrow[\varepsilon \downarrow 0]{d} \{X_*(t), t \geqslant 0\}.$$

With $\{Z^{\varepsilon}(t), t \geqslant 0\}$ the inverse extremal process, we also get

$$\{Z^{\varepsilon}(t),\ t \geqslant 0\} \xrightarrow[\varepsilon \downarrow 0]{d} \{Z(t),\ t \geqslant 0\}.$$

The limiting processes $\{X_*(t), t \geqslant 0\}$ and $\{Z(t), t \geqslant 0\}$ are defined by their finite-dimensional distributions. More precisely, for any $n \geqslant 1$, and $0 \leqslant x_1 < \cdots < x_n$, $0 \leqslant t_1 < \cdots < t_n$, these are now given by

$$\Pr(X_*(t_1) \leqslant x_1, \ldots, X_*(t_n) \leqslant x_n) = \prod_{k=1}^{n} \mathrm{e}^{-(t_k - t_{k-1}) x_k^{-\alpha} \exp \alpha v(\log x_k)} \tag{5.5}$$

and

$$\Pr(Z(t_1) > x_1, \ldots, Z(t_n) > x_n) = \prod_{k=1}^{n} \mathrm{e}^{-(x_k - x_{k-1}) t_k^{-\alpha} \exp \alpha v(\log t_k)}. \tag{5.6}$$

The extremal process is also a *Markov* process whose transition kernel and holding time distribution may easily be derived, whereas the inverse process has unstationary independent increments with *Lévy* characteristic which are also easily derivable.

We note in particular that

$$\Pr(X_*(t) \leqslant x) = \Pr(X_*(1) \leqslant x)^t \tag{5.7}$$

with now

$$\Pr(X_*(1) \leqslant x) = \mathrm{e}^{-x^{-\alpha} \exp \alpha v(\log x)}, \quad x > 0. \tag{5.8}$$

The extremal process $X_*(t)$ is semi-self-similar in the sense that, with $\sigma := c^{-\alpha}$, for any $\gamma_k := \sigma^k > 0$, $k \in \mathbb{Z}$ and any $0 \leqslant t_1 < \cdots < t_n$, we have

$$(X_*(\gamma_k t_1), \ldots, X_*(\gamma_k t_n)) \stackrel{d}{=} (\gamma_k^{1/\alpha} X_*(t_1), \ldots, \gamma_k^{1/\alpha} X_*(t_n)), \tag{5.9}$$

which may be checked from (5.5), exploiting the periodicity of function v. The semi-stability (scaling) exponent is $1/\alpha$, with $\alpha > 0$.

In a self-similar process with scaling exponent H, every change of time scale $\gamma > 0$ corresponds to a change of space scale γ^H: this is a "scale-invariance" property of the finite-dimensional distributions of the process. In a semi-stable process all change of time scale are not allowed; rather these are necessarily of the particular form $\gamma_k := \sigma^k > 0$, as $k \in \mathbb{Z}$, for some $\sigma > 0$.

Similarly, the inverse extremal process $\{Z(t), t \geqslant 0\}$ is found to be semi-self-similar with scaling exponent α.

Remark 4. Let $\sigma \geqslant 1$ and $c \in (0, 1]$ (or $\sigma \in (0, 1]$ and $c \geqslant 1$). Consider the class of positive random variable, say X, whose probability distribution function, say $F(x)$, satisfies the functional equation

$$F(x) = F(x/c)^{\sigma}, \quad x \geqslant 0. \tag{5.10}$$

These variables can be identified with the so-called max-semi-stable variables [10,11,15]. The class of solutions of (5.10) is then easily found to be

$$F(x) = \exp -x^{-\alpha} \mathrm{e}^{\alpha v(\log x)}, \tag{5.11}$$

where $\alpha > 0$ is defined through

$$\sigma c^{\alpha} = 1 \tag{5.12}$$

and where $v(z)$ is a bounded periodic (with period $-\log c$) function on the real line. As F must be the probability distribution function of some random variable the additional condition that $z - v(z)$ is a non-decreasing function has to be imposed, in such a way that the hazard function $x^{-\alpha} \mathrm{e}^{\alpha v(\log x)}$ be non-increasing with x. Note that *Fréchet* variables are recovered letting $\sigma \downarrow 1$ and $c \uparrow 1$ under the constraint (5.12).

From (5.7) and (5.8), in a semi-stable extremal process $\{X_*(t), t \geqslant 0\}$, the distribution of $X_*(1)$ is the one of a max-semi-stable variable.

5.2. Lévy semi-stable process

Fix $\alpha \in (0,1)$. Next consider the limiting sum process, obtained as the limit

$$\{X^{\varepsilon}(t), t \geqslant 0\} \xrightarrow[\varepsilon\downarrow 0]{d} \{X(t), t \geqslant 0\}$$

with

$$X^{\varepsilon}(t) := 0 \cdot \mathbf{1}(N^{\varepsilon}(t) = 0) + \left[\sum_{m=1}^{N^{\varepsilon}(t)} \chi_m^{\varepsilon}\right] \cdot \mathbf{1}(N^{\varepsilon}(t) > 0), \quad t \geqslant 0, \tag{5.13}$$

the associated compound *Poisson* process. From the *Poissonian* structure, we get

$$\mathbf{E}\,\mathrm{e}^{-\lambda\chi^{\varepsilon}(t)} = \mathrm{e}^{-m_{\varepsilon}t(1-\mathbf{E}\,\mathrm{e}^{-\lambda\chi^{\varepsilon}})}. \tag{5.14}$$

Now, with the *Lévy* spectral function

$$\pi(x) := -x^{-\alpha} \exp \alpha v(\log x) \tag{5.15}$$

we get

$$m_{\varepsilon}(1 - \mathbf{E}\,\mathrm{e}^{-\lambda\chi^{\varepsilon}}) = \int_{\varepsilon}^{+\infty} (1 - \mathrm{e}^{-\lambda x})\, \mathrm{d}\pi(x) \underset{\varepsilon\downarrow 0}{\rightarrow} \lambda^{\alpha}\mathrm{e}^{-\alpha\zeta(\log\lambda)} \tag{5.16}$$

for some periodic function $\zeta(q)$, $q := \log\lambda$, with period $\log c$ on the real line, characterized in the remark below. We therefore get the following definition:

A *Lévy* semi-stable process is the process $\{X(t), t \geqslant 0\}$ with *sii* such that

$$\mathbf{E}\,\mathrm{e}^{-\lambda X(t)} = \exp -t\lambda^{\alpha}\mathrm{e}^{-\alpha\zeta(\log\lambda)}, \quad t \geqslant 0. \tag{5.17}$$

From (5.16), the function $\zeta(q)$ is completely determined from the *Fourier* series expansion of function $v(z)$ [16].

We thus found that $\{X(t), t \geqslant 0\}$ is a process with stationary independent increments such that

$$\mathbf{E}\,\mathrm{e}^{-\lambda X(t)} = \mathrm{e}^{-t\lambda^{\alpha}\mathrm{e}^{-\alpha\zeta(\log\lambda)}} = [\mathbf{E}\,\mathrm{e}^{-\lambda X(1)}]^{t}. \tag{5.18}$$

From this expression, the limiting sum process $X(t)$ is semi-self-similar in the sense that for any $\gamma_k := \sigma^k > 0$, $k \in \mathbb{Z}$, and any $0 \leqslant t_1 < \cdots < t_n$, we have

$$(X(\gamma_k t_1), \ldots, X(\gamma_k t_n)) \stackrel{d}{=} (\gamma_k^{1/\alpha}X(t_1), \ldots, \gamma_k^{1/\alpha}X(t_n)), \tag{5.19}$$

which may be checked from (5.19), now exploiting the periodicity of function ζ. The semi-self-similarity (scaling) exponent is $1/\alpha$, with $\alpha \in (0,1)$.

Remark 5. Let $\sigma \geqslant 1$ and $c \in (0,1]$ or $\sigma \in (0,1]$ and $c \geqslant 1$. Consider the class of positive random variable whose LST, say $\varphi(\lambda)$, satisfies the functional equation

$$\varphi(\lambda) = \varphi(c\lambda)^{\sigma}, \quad \lambda \geqslant 0. \tag{5.20}$$

These variables can be identified with the so-called positive semi-stable variables [16,20,22,24,28]. The class of solutions of (5.20) is then easily found to be, formally,

$$\varphi(\lambda) = \exp -\lambda^{\alpha}\mathrm{e}^{\alpha\zeta(\log\lambda)}, \tag{5.21}$$

where $\alpha > 0$ is defined through

$$\sigma c^{\alpha} = 1 \tag{5.22}$$

and where $\zeta(q)$, $q := \log\lambda$, $\lambda > 0$, is a periodic (with period $\log c$) function on the real line. From (5.18) and (5.21), in a semi-stable sum process $\{X(t), t \geqslant 0\}$, the distribution of $X(1)$ is the one of a semi-stable variable. Note that in (5.21), the parameter necessarily lies in the interval $(0,1)$ if $\varphi(\lambda)$ is to be LST of some probability distribution.

6. *Lévy* and *Fréchet* Geometric semi-stable processes

In this section, we introduce a related class of semi-stable processes, namely the class of Geometric *Lévy* and Geometric *Fréchet* semi-stable processes.

6.1. Semistable processes in random Exponential time: subordination

Let $\{X(t), t \geqslant 0\}$ be a *Lévy* semi-stable process with exponent $\alpha \in (0,1)$, independent of the Exponential process defined $\{E(t), t > 0\}$ above.

Consider next the "subordinated" process $\{G(t) := X(E(t)), t > 0\}$. This process has independent unstationary increments. Besides, as it can easily be checked

$$\mathbf{E}\,\mathrm{e}^{-\lambda G(t)} = \frac{1}{1 + t\lambda^{\alpha}\mathrm{e}^{\alpha\zeta(\log\lambda)}}. \tag{6.1}$$

As a composition of a self-similar process and of a semi-self-similar process, this process is itself semi-self-similar. Its scaling exponent is $1/\alpha$, in the sense that, with $\gamma_k = \sigma^k, k \in \mathbb{Z}$,

$$(G(\gamma_k t_1), \ldots, G(\gamma_k t_n)) \stackrel{d}{=} (\gamma_k^{1/\alpha} G(t_1), \ldots, \gamma_k^{1/\alpha} G(t_n)). \tag{6.2}$$

The process $\{G(t), t \geqslant 0\}$ is known as the Geometric *Lévy* semi-stable process.

Let $\{X_*(t), t > 0\}$ be a *Fréchet* semi-stable process with exponent $\alpha \in (0,1)$, independent of $\{E(t), t > 0\}$.

Consider next the subordinated process $\{G_*(t) := X_*(E(t)), t > 0\}$. Clearly, this process is *Markovian*. We note in particular that

$$\Pr(G_*(t) \leqslant x) = \int_0^{\infty} \mathrm{e}^{-sx^{-\alpha}} \frac{1}{t} \mathrm{e}^{-st^{-1}}\, \mathrm{d}s = \frac{1}{1 + tx^{-\alpha}\mathrm{e}^{\alpha\nu(\log x)}}. \tag{6.3}$$

As a result, it is also semi-stable with exponent $1/\alpha$, in the sense that

$$(G_*(t)(\gamma_k t_1), \ldots, G_*(\gamma_k t_n)) \stackrel{d}{=} (\gamma_k^{1/\alpha} G_*(t_1), \ldots, \gamma_k^{1/\alpha} G_*(t_n)). \tag{6.4}$$

The process $\{G_*(t), t \geqslant 0\}$ is known as the Geometric *Fréchet* semi-stable process.

Consider finally the output process

$$L_*(t) = \alpha \log G_*(t). \tag{6.5}$$

Its finite-dimensional distributions may be computed from the ones of the geometric *Fréchet* semi-stable process. In particular, from (6.3) and (6.5)

$$\Pr(L_*(t) \leqslant x) = \frac{1}{1 + t\mathrm{e}^{-x}\mathrm{e}^{\alpha\nu(x/\alpha)}}. \tag{6.6}$$

We note that $\Pr(L_*(1) \leqslant x) = (1 + \mathrm{e}^{-x}\mathrm{e}^{\alpha\nu(x/\alpha)})^{-1}, x \in (-\infty, +\infty)$. When $\nu = 0$, this cumulative distribution function is the one of a logistic distribution. Thus $L_*(t)$ is identified with the logistic extremal process. It is a semi-self-similar process of the translational type in the sense that, for any $\gamma_k := \sigma^k > 0$, $k \in \mathbb{Z}$,

$$(Y_*(\gamma_k t_1), \ldots, Y_*(\gamma_k t_n)) \stackrel{d}{=} (Y_*(t_1) + \log\gamma_k, \ldots, Y_*(t_n) + \log\gamma_k). \tag{6.7}$$

Remark 6. Let $\sigma \geqslant 1$ and $c \in (0,1]$. Let $N(\sigma)$ be a geometric discrete random variable on the positive integers $\{1, \ldots, n, \ldots\}$ with mean value σ. Let $\phi_\sigma(u) := \mathbf{E}u^{N(\sigma)}$ be the probability generating function of $N(\sigma)$. Under our geometric hypothesis, it is

$$\phi_\sigma(u) = \frac{1}{1 + \sigma(u^{-1} - 1)}, \quad u \in [0,1].$$

Consider now the class of positive random variable whose probability distribution function, say $F(x)$, satisfies the functional equation

$$F(x) = \phi_\sigma(F(x/c)), \quad x \geqslant 0. \tag{6.8}$$

This functional equation extends the one in (5.10) in the sense that the intensity parameter σ is now allowed to be random, substituting $N(\sigma)$ to σ. Note however that $\mathbf{E}N(\sigma) = \sigma$.

These variables are to be identified with the so-called geometric max-semi-stable variables. The class of solutions of (6.8) is then easily found to be

$$F(x) = \frac{1}{1 + x^{-\alpha} e^{\alpha v(\log x)}}, \tag{6.9}$$

where $\alpha > 0$ is defined through $\sigma c^{\alpha} = 1$. From (6.3) and (6.9), in a geometric extremal process $\{G_*(t), t \geqslant 0\}$, the distribution of $G_*(1)$ is the one of a geometric-max-semi-stable variable. If function $v = 0$, random variables with cumulative distribution function: $F(x) = (1 + x^{-\alpha})^{-1}$ are the geometric-max-stable.

In a similar way, extending (5.20), consider the class of positive random variables whose LST, say $\varphi(\lambda)$, satisfies the functional equation

$$\varphi(\lambda) = \phi_{\sigma}(\varphi(c\lambda)), \quad \lambda \geqslant 0. \tag{6.10}$$

These variables can be identified with the so-called positive geometric-semi-stable variables (for sums). The class of solutions of (6.10) is then easily found to be

$$\varphi(\lambda) = \frac{1}{1 + \lambda^{\alpha} e^{\alpha \zeta(\log \lambda)}}, \tag{6.11}$$

where $\alpha \in (0, 1)$ is defined through $\sigma c^{\alpha} = 1$. If the function ζ is constant: $\zeta(q) := \zeta$, and if $a := e^{\alpha\zeta}$, random variables with LST: $\varphi(\lambda) = (1 + a\lambda^{\alpha})^{-1}$ are thus the previous geometric-stable variables.

From (6.1) and (6.11), in a geometric semi-stable sum process $\{G(t), t \geqslant 0\}$, the distribution of $G(1)$ is the one of a geometric semi-stable variable.

7. Concluding remarks

The strong connections between *Lévy*-stable, *Fréchet* and inverse *Fréchet* has been displayed. The Geometric version of such processes have been investigated. Some of their properties have been listed which illustrates that self-similarity may be compatible with different processes' structures. A larger class of related semi-self-similar processes is also investigated. All are *Markov* examples of concrete interest of self-similar and semi-self-similar processes that are not necessarily with stationary and independent increments nor with independent increments. This illustrates that these notions are much more general than the ones of *Lévy*'s stability and semi-stability.

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