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ON LÉVY STABLE AND SEMISTABLE DISTRIBUTIONS

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Abstract

This work emphasizes the special role played by *semistable* and *log-semistable* distributions as relevant statistical models of various observable and "internal" variables in physics. Besides of their representation, some of their remarkable properties (chiefly *semi-self-similarity*) are displayed in some detail. One of their characteristic features is a log-periodic variation of the scale parameter which appears in the standard *Lévy α-stable* distributions whose *Fourier* representations are re-derived in a self-contained way.

1. INTRODUCTION

This work emphasizes the special role played by *semistable* and *log-semistable* distributions as relevant statistical models of various observable and "internal" variables arising from the natural sciences such as, say, hydrology, geophysics, finance, etc. The fitting of such distributions to real-world problems is not addressed here and is postponed to a future work, rather the chief objective is to discuss the very particular statistical status and properties; which such distributions seem to entail. On this basis, we tried to justify why *semistability* should play a central role in the modeling of random observable. This work is then organized as follows.

In Sec. 2, we first address the well-known problem which consists in modeling observed random events out of the celebrated *Lévy* α-stable distributions, which are known to embody the only possible limiting distributions for sums of independent and identically distributed (*iid*) random variables.[1] Some of their remarkable properties are briefly discussed, focusing in particular on the one of their *stability* under the operation of "sum" and *self-similarity*. The motivation behind this section is not in the final results (which are very standard), rather in the way these results are derived; indeed, proceeding in this way extends easily to representation questions of the larger class of *semistable self-similar* laws.

In Sec. 3, we discuss a concept whose statistical insight is indeed deeper than the one of *stability*, namely the one of *semistability*, leading to a larger class of *infinitely divisible* statistical distributions.[2–5] These can be defined as the fixed point of some transformation on their *Laplace-Stieltjes* or *Fourier* transform (FT) which basically reflects the statistical *self-similarity* properties of their solutions. More precisely, *semistable* observable, as random events, are identified with the ones whose FT (or characteristic function), say ϕ, satisfies a functional equation of the form

$$\phi(\lambda) = e^{i\lambda\beta}\phi(c\lambda)^{\gamma}$$

for some $c > 0$, $\beta \in \mathbf{R}$, $\gamma > 1$. Forcing $\beta = 0$ in this functional equation yields solutions whose main feature identifies with the notion of *semi-self-similarity*. Allowing $\beta \neq 0$ identifies with a notion of *semi-self-similarity* in the broad sense (or *semistability*), allowing for shifts to explain the observable.

We exhibit all possible solutions to the above functional equation, extending (and actually including) the *stable* distributions. The support of these (two-sided) distributions is the whole real line; there exists a one-sided version of *semistable* (and *stable*) distributions with the half-line as support which are essential in the design of their two-sided extension. A characteristic feature of *semistable* distributions is that their scale parameters are no longer constant as for their *stable* restriction, but rather allowed to vary in a log-periodic fashion. In our interpretation, a log-periodic scale parameter is therefore basically the signature of statistical *semi-self-similarity*. These distributions stand as appealing candidates to model *observed* random events in the natural sciences, precisely because of their *semi-self-similarity*. In this interpretation, the observed "global" random event interprets as the sum of a *Poissonian* number of "local" events, in other words, as a clustering of "micro-events."

What the functional equation adds to this is that this global event could as well result from *more* local events but with reduced shifted amplitudes. There is some incertitude on the way the final global result may be produced in that the exact scale, intensity and location of the observable are undetermined.

It is a very common feature in Physics that *observable* are modeled through the logarithms of other quantities with much more "appealing" meaning, such as the "energy" or intensity of some underlying phenomenon. The distinctive feature of the logarithmic scale for observable is that it measures the distance between two intensities through their ratio rather than their difference; this transformation thus supplies a discrimination power between two physical signals which is insensitive to their absolute intensities in that it essentially deals with their ratio. In our context, this means that *semistable* variables are to be considered as the observable of some "hidden" physical phenomenon, which therefore proves itself *log-semistable*. Consequently, it is argued that *log-semistable* distributions should stand for most relevant statistical models for the intensity of random events measured in logarithmic scale.

In Sec. 4, we thus focus on *log-semistable* models. In these models, the tails of the energy variable are extremely heavy, i.e. with tail index zero, whereas the ones of their logarithm (the associated observable) are "only" of power-law type, emphasizing the fact that observable generically exhibit tails much thinner than the ones of the underlying hidden energy variable. This fact and the logarithmic scale arguments suggest that when dealing with a sequence of intensities, if one is to understand an "average" event, one should rather use geometrical averaging since the standard arithmetic average may be extremely ill-defined.

2. STABLE MODELS FOR RANDOM EVENTS

We first recall some salient facts concerning stable random variables (see Uchaikin and Zolotarev (1999)[1] and Sato (1999)[6] for a recent overview of these problems). The adjective stable refers to the well-known identity in distribution

$$\text{for any } c_1 > 0,\ c_2 > 0, \quad \text{there exists } c > 0,$$
$$b \in \mathbb{R}:\ c_1 X_1 + c_2 X_2 \stackrel{d}{=} cX + b$$

with $X \stackrel{d}{=} X_1 \stackrel{d}{=} X_2$ mutually independent. If $b = 0$, X is said to be strictly stable. These models are known[6–8] to belong to a subclass of infinitely divisible random variables which are those for which the nth root of their characteristic function still is a characteristic function for any $n \geq 1$.[9,11]

2.1 One-Sided Stability

A random variable is said to be one-sided if the support of its distribution is a half-line. We now come to one-sided stability.

2.1.1 *One-sided strictly stable laws*

A random variable X^+ is a one-sided strictly *Lévy*-stable variable with index $\alpha \in (0, 1)$, if it is positive and if the *Laplace-Stieltjes transform* (*LST*) of its distribution is (with $\mathbf{E}$ the symbol for mathematical expectation)

$$\begin{aligned}\varphi_{X^+}(p) &:= \mathbf{E}e^{-pX^+} \\ &= \exp -s_0 p^\alpha\,, \quad s_0 > 0,\ p \geq 0\,. \end{aligned} \tag{1}$$

One may easily check that such random variables are strictly stable and infinitely divisible. Note that when $\alpha > 1$, $\varphi_{X^+}(p)$ is not the *LST* of a positive random variable by *Bernstein*'s theorem.[7]

Define the monotone increasing *Lévy* spectral function

$$\pi(x) := -\kappa x^{-\alpha}\,, \quad \kappa > 0\,, \quad x \in (0, +\infty)\,. \tag{2}$$

This function induces a positive *Radon* measure $d\pi(x)$, (known as *Lévy* measure), defined by the *Stieltjes* integral

$$\pi(x_2) - \pi(x_1) := \int_{x_1}^{x_2} d\pi(x)\,, \quad x_2 > x_1 > 0\,.$$

Now, with $s_0 := \kappa\Gamma(1-\alpha)$, the following identity holds (which may be checked upon integrating by parts)

$$\int_0^{+\infty} (1 - e^{-px})d\pi(x) = s_0 p^\alpha \tag{3}$$

in such a way that the *Lévy-Khintchine* representation holds

$$\varphi_{X^+}(p) = \exp - \int_0^{+\infty} (1 - e^{-px})d\pi(x)\,.$$

Note that the random variable $-X^+$ is a negative one-sided strictly stable variable.

2.1.2 *One-sided strictly stable laws as limit compound Poisson distributions*

In (3), the measure $d\pi(x)$ has infinite total mass, due to the algebraic divergence of its density

$$d\pi(x)/dx := \kappa\alpha x^{-(1+\alpha)}$$

in the vicinity of zero.

As $d\pi(x)$ is not a probability measure, a one-sided *Lévy* variable, for which (1) holds, is not a compound *Poisson* variable, i.e. a *Poisson* sum of *iid positive* jumps; rather, it can be obtained from a "coarse" compound *Poisson* variable X^ε in the limit $\varepsilon \downarrow 0$.

Let indeed $\varepsilon > 0$; consider the compound *Poisson* variable X^ε with intensity m^ε and with jumps with normalized *truncated* probability density

$$f_\varepsilon(x) = \frac{\kappa\alpha}{m^\varepsilon} \cdot x^{-(1+\alpha)} \cdot \mathbf{1}\ (x > \varepsilon)\,. \tag{4}$$

We note that this is a *Pareto* distribution in the heavy-tailed class with tail index α.

From (4), the normalization constant is easily obtained. It is

$$m^\varepsilon = \int_\varepsilon^{+\infty} d\pi(x) = \kappa\varepsilon^{-\alpha} \tag{5}$$

and m^ε tends to infinity as ε tends to zero. Next, we form the quantity $\mathbf{E}e^{-pX^\varepsilon}$. From the definition of a compound *Poisson* variable, it is

$$\mathbf{E}e^{-pX^\varepsilon} = e^{-m^\varepsilon\left(1-\int_0^{+\infty} e^{-px} f_\varepsilon(x)dx\right)}\,. \tag{6}$$

Now,

$$\begin{aligned} &m^\varepsilon\left(1 - \int_0^{+\infty} e^{-px} f_\varepsilon(x)dx\right) \\ &\quad = \int_\varepsilon^{+\infty} (1 - e^{-px})d\pi(x) \underset{\varepsilon\downarrow 0}{\sim} s_0 p^\alpha \end{aligned} \tag{7}$$

which is consistent with (1).

Thus, X^+ defined from (1) is the limiting compound *Poisson* variable X^ε as $\varepsilon \downarrow 0$ and is indeed an infinitely divisible variable.

Remark 1. Let us now recall,[7] that a distribution is said to be *regularly varying* if there exists some finite strictly positive constant α (the tail exponent) such that

$$\Pr(X > x) = x^{-\alpha} L(x) \quad (8)$$

where L is some function with slow variation, i.e. such that for all strictly positive t:

$$\lim_{x \uparrow +\infty} \frac{L(tx)}{L(x)} = 1 . \quad (9)$$

Such distributions have no moments of order larger than α.

Clearly, for the *Lévy* variable X^+, $\varphi_{X^+}(p) \underset{p\downarrow 0}{\sim} 1 - s_0 p^\alpha$, so that the *Tauberian* theorem applies[7] and

$$\Pr(X^+ > x) \underset{x\uparrow+\infty}{\sim} \kappa x^{-\alpha} . \quad (10)$$

Thus, the *Lévy* distribution of X^+ is heavy-tailed, with tail exponent α; the variable X^+ only possesses moments of order strictly less than α. Note that as $\alpha \in (0, 1)$, X^+ does not even have a mean value, i.e. $\mathbf{E}(X^+) = +\infty$.

2.1.3 *Shifted one-sided stable laws*

Actually, the class of all one-sided *Lévy* distributions can be obtained while allowing a shift of one-sided strictly stable variables. Let $x^+ \in \mathbf{R}$. Consider the shifted variable $\tilde{X}^+ := X^+ + x^+$. From (1), it now admits the *LST*

$$\varphi_{\tilde{X}^+}(p) = \exp -(p x^+ + s_0 p^\alpha) , \quad p \geq 0 \quad (11)$$

which is the one-sided *Lévy* "*stable*" model with support the half line $(x^+, +\infty)$, unbounded to the right. Note that the random variable $-\tilde{X}^+$ is also a shifted one-sided stable variable but with support now unbounded to the left.

2.2 Two-Sided Stability

We shall now extend this construction to the whole real line. We shall distinguish three cases, depending on the range of exponent α.

2.2.1 *Exponent* $\alpha \in (0, 1)$

Consider two independent positive one-sided *Lévy* variables, say X_1^+ and X_2^+, with respective *Lévy* spectral function

$$\pi_l(x) = -\kappa_l x^{-\alpha} , \quad \kappa_l > 0 , \quad l = 1, 2, \, x > 0 . \quad (12)$$

Define next the full *Lévy* spectral function on the real line (except for zero) as

$$\pi(x) = \pi_1(x) \mathbf{1}\ (x > 0) - \pi_2(-x) \mathbf{1}\ (x < 0) . \quad (13)$$

This spectral function is monotone increasing, with $\pi(-\infty) = \pi(+\infty) = 0$, $\pi(0^-) = +\infty$ and $\pi(0^+) = -\infty$. It thus induces a positive *Lévy* measure $d\pi(x)$, with support $(-\infty, 0) \cup (0, +\infty)$.

Next consider the real-valued random variable

$$X = X_1^+ - X_2^+ . \quad (14)$$

Its characteristic function which is the *Fourier* transform (FT) of its distribution is

$$\phi_X(\lambda) := \mathbf{E} e^{i\lambda X} = \varphi_{X_1^+}(-i\lambda) \varphi_{X_2^+}(i\lambda) . \quad (15)$$

In more explicit form, we get from Subsec. 2.1

$$\begin{aligned} \phi_X(\lambda) &= \exp - \left[\int_{-\infty}^{+\infty} (1 - e^{i\lambda x}) d\pi(x) \right] \\ &= \exp -\Gamma(1-\alpha)[\kappa_1(-i\lambda)^\alpha + \kappa_2 (i\lambda)^\alpha] . \quad (16) \end{aligned}$$

Alternatively, using the identity $i\lambda = |\lambda| \exp \cdot i \,\mathrm{sign}(\lambda) \frac{\pi}{2}$, we easily get

$$\phi_X(\lambda) = \exp -s|\lambda|^\alpha \left(1 - i\rho \,\mathrm{sign}(\lambda) \tan \frac{\pi\alpha}{2} \right) \quad (17)$$

where

$$s = \Gamma(1-\alpha)(\kappa_1 + \kappa_2) \cos \frac{\pi\alpha}{2} > 0 ,$$

and

$$\rho = \frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} \in [-1, +1] . \quad (18)$$

The parameter ρ is thus a skewness parameter. If $\kappa_1 = \kappa_2 = \kappa$, then $\rho = 0$ and

$$s = 2 \cos \frac{\pi\alpha}{2} \kappa \Gamma(1-\alpha) = 2 \cos \frac{\pi\alpha}{2} s_0 \quad (19)$$

characterizing symmetric *Lévy* stable variables.

Let $x \in \mathbf{R}$. Consider the shifted variable $\tilde{X} := X + x$. From (17), the shifted variable now admits the *FT*

$$\phi_{\tilde{X}}(\lambda) = \exp\, i\lambda x - s|\lambda|^\alpha \left(1 - i\rho\, \mathrm{sign}(\lambda)\, \tan\frac{\pi\alpha}{2}\right) \tag{20}$$

with shift x.

Remark 2. Under the hypothesis $\alpha \in (0,\,1)$, the number $x_1 := \int_{|x|\leq 1} x d\pi(x)$ is finite, so that, from (16), we get the *Lévy-Khintchine* representation

$$\phi_X(\lambda) = \exp\, i\lambda x_1 - \left[\int_{-\infty}^{+\infty} (1 + i\lambda x \mathbf{1}\,(|x| \leq 1) - e^{i\lambda x}) d\pi(x)\right]. \tag{21}$$

Thus, with $\tilde{x} := x + x_1$, the *FT* of $\tilde{X} = X + x$ also admits the *Lévy-Khintchine* representation

$$\phi_{\tilde{X}}(\lambda) = \exp\, i\lambda\tilde{x} - \left[\int_{-\infty}^{+\infty} (1 + i\lambda x \mathbf{1}\,(|x| \leq 1) - e^{i\lambda x}) d\pi(x)\right] \tag{22}$$

which is also

$$\phi_{\tilde{X}}(\lambda) = \exp\, i\lambda(\tilde{x} - x_1) - s|\lambda|^\alpha \left(1 - i\rho\, \mathrm{sign}(\lambda) \tan\frac{\pi\alpha}{2}\right). \tag{23}$$

By (21), (22) and (23), we note that the *Lévy-Khintchine* representation of strictly α-*Lévy* stable variables on the real line are obtained while taking $\tilde{x} = x_1$.

2.2.2 *Exponent* $\alpha \in (1,\,2)$

In this parameter range, the identity (3) is invalid. With $p \geq 0$, we shall rather use the identity (which may be checked upon first integrating by parts and then applying identity (3)

$$\int_0^{+\infty} (1 - px - e^{-px}) d\pi_l(x) = \kappa_l \Gamma(1-\alpha) p^\alpha\,, \quad l = 1,\,2 \tag{24}$$

with now $\Gamma(1-\alpha) < 0$ (Lavrentiev and Chabat (1972),[10] p. 590). As a result, with function π defined as in (12) and (13), viz $\pi(x) = \pi_1(x)\mathbf{1}\,(x > 0) - \pi_2(-x)\mathbf{1}\,(x < 0)$

$$\begin{aligned}\int_{-\infty}^{+\infty} (1 + i\lambda x - e^{i\lambda x}) d\pi(x) &= \Gamma(1-\alpha)[\kappa_1(-i\lambda)^\alpha + \kappa_2(i\lambda)^\alpha] \\ &= s|\lambda|^\alpha \left(1 - i\rho\, \mathrm{sign}(\lambda) \tan\frac{\pi\alpha}{2}\right)\end{aligned} \tag{25}$$

where again

$$s = \Gamma(1-\alpha)(\kappa_1 + \kappa_2)\, \cos\frac{\pi\alpha}{2} > 0$$

and
$$\rho = \frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} \in [-1,\, +1] \tag{26}$$

observing that $\cos\frac{\pi\alpha}{2} < 0$ in the parameter range $\alpha \in (1,\,2)$.

Note that in this parameter range, $x_1 := -\int_{|x|>1} x d\pi(x) < \infty$. Hence,

$$\int_{-\infty}^{+\infty} (1 + i\lambda x \mathbf{1}\,(|x| \leq 1) - e^{i\lambda x}) d\pi(x) = \int_{-\infty}^{+\infty} (1 + i\lambda x - e^{i\lambda x}) d\pi(x) + i\lambda x_1\,. \tag{27}$$

Let $\tilde{x} \in \mathbf{R}$. Define the *FT* of a shifted *Lévy* variable, as in the case $\alpha \in (0,\,1)$, by

$$\phi_{\tilde{X}}(\lambda) = \exp\, i\lambda\tilde{x} - \int_{-\infty}^{+\infty} (1 + i\lambda x \mathbf{1}\,(|x| \leq 1) - e^{i\lambda x}) d\pi(x)\,. \tag{28}$$

From (25) and (27), we get

$$\phi_{\tilde{X}}(\lambda) = \exp\, i\lambda(\tilde{x} - x_1) - s|\lambda|^\alpha \left(1 - i\rho\, \mathrm{sign}(\lambda)\, \tan\frac{\pi\alpha}{2}\right). \tag{29}$$

Taking $\tilde{x} - x_1 = 0$ yields the *FT* of a strictly α-*Lévy* stable variable on the real line, say X. In this case, it has no shift in its *FT*. It is

$$\phi_X(\lambda) = \exp -s|\lambda|^\alpha \left(1 - i\rho\, \mathrm{sign}(\lambda)\, \tan\frac{\pi\alpha}{2}\right). \tag{30}$$

2.2.3 *Exponent* $\alpha = 1$

In this critical case, with $p \geq 0$, the following identity is of some use

$$\int_0^{+\infty} (1 - p \sin\, x - e^{-px}) \frac{1}{x^2}\, dx = -p\, \log\, p\,. \tag{31}$$

As a result, with function π defined in (13), and with $\pi_l(x) = -\kappa_l x^{-1}$, $\kappa_l > 0$, $x > 0$, $l = 1, 2$, we get as in (Feller (1971),[7] p. 569)

$$\int_{-\infty}^{+\infty} (1 + i\lambda \sin x - e^{i\lambda x}) d\pi(x)$$
$$= \kappa_1 i\lambda \log(-i\lambda) - \kappa_2 i\lambda \log(i\lambda)$$
$$= \frac{\pi}{2}(\kappa_1 + \kappa_2)|\lambda| \left(1 + i\rho \frac{2}{\pi} \operatorname{sign}(\lambda) \log|\lambda|\right). \tag{32}$$

Observing that, with $x_1 := \int_{-\infty}^{+\infty} (x\mathbf{1}\ (|x| \le 1) - \sin x) d\pi(x) < \infty$, we have

$$\int_{-\infty}^{+\infty} (1 + i\lambda \sin x - e^{i\lambda x}) d\pi(x)$$
$$= \int_{-\infty}^{+\infty} (1 + i\lambda x\mathbf{1}\ (|x| \le 1) - e^{i\lambda x}) d\pi(x) - i\lambda x_1 . \tag{33}$$

Finally, we obtain the FT for the shifted *Lévy* variable in the case $\alpha = 1$

$$\phi_{\tilde{X}}(\lambda) = \exp i\lambda(\tilde{x} - x_1) - \frac{\pi}{2}(\kappa_1 + \kappa_2)|\lambda| \times \left(1 + i\rho \frac{2}{\pi} \operatorname{sign}(\lambda) \log|\lambda|\right). \tag{34}$$

Taking $\rho = 0$ yields the FT of a 1-*Lévy* strictly stable variable on the real line, say X, now possibly with a shift component in its FT, and

$$\phi_X(\lambda) = \exp i\lambda(\tilde{x} - x_1) - \frac{\pi}{2}(\kappa_1 + \kappa_2)|\lambda|. \tag{35}$$

2.2.4 *Conclusions*

Putting all this together, with π the full *Lévy* spectral function for jumps defined by (12) and (13), one may say that two-sided *Lévy* stable variables are the ones whose FT are given by

$$\phi_{\tilde{X}}(\lambda) = \exp i\lambda\tilde{x} - \int_{-\infty}^{+\infty} (1 + i\lambda x\mathbf{1}\ (|x| \le 1) - e^{i\lambda x}) d\pi(x) \tag{36}$$

with second characteristic function

$$\int_{-\infty}^{+\infty} (1 + i\lambda x\mathbf{1}\ (|x| \le 1) - e^{i\lambda x}) d\pi(x) = i\lambda x_1 + s|\lambda|^{\alpha}(1 - i\rho \operatorname{sign}(\lambda) h_{\alpha}(\lambda)). \tag{37}$$

Here, $x_1 := \int_{|x|\le 1} x d\pi(x)$ as $\alpha \in (0, 1)$, $x_1 := -\int_{|x|>1} x d\pi(x)$ as $\alpha \in (1, 2)$, and $x_1 := \int_{-\infty}^{+\infty} \cdot (x\mathbf{1}\ (|x| \le 1) - \sin x) d\pi(x)$, as $\alpha = 1$.

In addition, $h_{\alpha}(\lambda) = \tan \frac{\pi\alpha}{2}$ as $\alpha \in (0, 1) \cup (1, 2)$, and $h_{\alpha}(\lambda) = -\frac{2}{\pi} \log|\lambda|$ as $\alpha = 1$.

In any case, for any $\alpha \in (0, 2)$, $\rho = (\kappa_1 - \kappa_2)/(\kappa_1 + \kappa_2)$ is a skewness coefficient. The scale parameter s is $s = \Gamma(1 - \alpha) \cos \frac{\pi\alpha}{2}(\kappa_1 + \kappa_2)$ if $\alpha \in (0, 1) \cup (1, 2)$ and $s = \pi/2(\kappa_1 + \kappa_2)$, if $\alpha = 1$.

Note finally that the case $\alpha = 2$ (corresponding to the normal law with variance 2κ, $\kappa > 0$) for which,

$$\phi_{\tilde{X}}(\lambda) = \exp i\lambda\tilde{x} - \kappa\lambda^2, \quad \phi_X(\lambda) = \exp -\kappa\lambda^2 \tag{38}$$

is standard and is to be added. Random variables whose *Fourier* transforms are given by (17), (30) and (35) are known as two-sided strictly α-stable *Lévy* variables. If in addition, $\rho = 0$, we get the class of symmetric two-sided α-stable *Lévy* variables.

Random variables whose *Fourier* transforms are given by (23), (29) and (34) are known as shifted two-sided α-stable *Lévy* variables.

2.3 Stable Models as Limit Laws in Statistics

The distributions characterized by (1), (11), (17), (23), (29), (30), (34) and (35) are all *stable* in the following sense. Let X be any variable with such distribution. Then, for any $n \ge 1$, there exists two sequences $x_n \in \mathbf{R}$, $\sigma_n > 0$ such that, with $X_m \overset{d}{=} X$, $m = 1, \ldots, n$, an independent sequence

$$X \overset{d}{=} \sum_{m=1}^{n} \frac{X_m - x_n}{\sigma_n}. \tag{39}$$

Here the symbol $\overset{d}{=}$ means that the random variables share the same distributions. This class is a proper subclass of the one of infinitely divisible (ID) variables X for which, for any $n \ge 1$, with $\mathcal{X}_{m,n} \overset{d}{=} \mathcal{X}_n$, $m = 1, \ldots, n$

$$X \overset{d}{=} \sum_{m=1}^{n} \mathcal{X}_{m,n}. \tag{40}$$

A characteristic criterion for ID random variables X is thus that for any $n \ge 1$, $\phi_X(\lambda)^{1/n}$ must be the FT of a probability distribution (the one of $\mathcal{X}_n$); alternatively, for any $t > 0$, $\phi_X(\lambda)^t$ must be a FT, i.e. the characteristic function of some random variable.

Turning back to *stable* variables, a consequence of their properties is that they appear as all possible nondegenerate limit laws for sums in the following sense: if X is *stable*, then there exists an *iid* sequence $\mathcal{X}_m \stackrel{d}{=} \mathcal{X}$, $m \geq 1$ and two sequences $x_n \in \mathbf{R}$, $\sigma_n > 0$ such that

$$\sum_{m=1}^{n} \frac{\mathcal{X}_m - x_n}{\sigma_n} \stackrel{d}{\to} X \text{ as } n \uparrow +\infty . \tag{41}$$

The variable $\mathcal{X}$ is said to belong to the domain of attraction (DA) of X (note from (39) that X itself belongs to its own DA).

Thus, *stable* distributions derive their importance from the fact that they are the limit laws for sums of *iid* random variables $\mathcal{X}_m$, $m \geq 1$, after a convenient location-scale transform, which can be found in Ref. 1, for example.

Remark 3. Besides the stability property, stable laws are interesting in practice because of their statistical *self-similarity.* Indeed, when $\alpha \in (0, 2)$, the strictly stable variables X share the additional self-similarity property: for any $\varsigma > 0$

$$X(\varsigma) \stackrel{d}{=} \varsigma^{1/\alpha} X . \tag{42}$$

Here X is a random variable with FT $\phi_X(\lambda)$ given either by (17), (30) and (35). With $\varsigma > 0$, the variable $X(\varsigma)$ is defined as the variable whose FT is $\phi_{X(\varsigma)}(\lambda) = \phi_X(\lambda)^\varsigma$, raising ϕ_X to the power ς (this property is the one of *infinite divisibility* of X). The variable X is said to be statistically *self-similar*, with exponent $1/\alpha$: strictly α-stable variables are self-similar. Broadening the definition of self-similarity to

$$\tilde{X}(\varsigma) \stackrel{d}{=} \varsigma^{1/\alpha}(\tilde{X} - \beta) \tag{43}$$

for some real-valued constant β, one may check that all α-stable *Lévy* variables $\tilde{X}$ whose FT are given by (23), (29) and (34) are all self-similar in this broad sense, for any $\alpha \in (0, 2)$.

2.4 Symmetric Lévy Stable Distributions as Subordinates

Before proceeding, we recall an interesting connection between symmetric α-stable variables and one-sided *Lévy* stable ones.

Let $\alpha \in (0, 2)$ and X^+ be a one-sided *Lévy*-stable variable with index $\alpha/2 \in (0, 1)$ hence with LST

$$\varphi_{X^+}(p) := \mathbf{E}e^{-pX^+} = \exp\, -s_0 p^{\alpha/2}$$
$$s_0 = \Gamma(1 - \alpha/2)\kappa , \quad p \geq 0 .$$

Let $(B(t),\, t \geq 0)$, $B(0) = 0$, be a centered *Brownian* motion with variance 2κ and standard deviation $\sqrt{2\kappa}$, $\kappa > 0$. As a result,

$$\mathbf{E}e^{i\lambda B(t)} = \exp\, -t\kappa\lambda^2 .$$

Assume X^+ and $(B(t),\, t \geq 0)$ are independent and consider the random variable

$$X = B(X^+) .$$

After conditioning, we get

$$\phi_X(\lambda) := \mathbf{E}e^{i\lambda X} = \mathbf{E}\, \exp\, -\kappa\lambda^2 X^+$$
$$= \exp\, -s_0 \kappa^{\alpha/2} |\lambda|^\alpha$$

which is the FT of a symmetric two-sided α-stable *Lévy* variable with scale parameter $s = s_0\kappa^{\alpha/2}$.

3. SEMI-SELF-SIMILARITY AND THE SEMISTABLE MODELS

We now discuss a concept whose generality is larger than the one of *stability*, namely the one of *semistability.*

Just like for stable laws, there are one-sided and two-sided *semistable* laws.

One-sided (unbounded to the right) *semistable* laws are identified with the ones whose *Laplace-Stieltjes* transform satisfies a functional equation of the form

$$\varphi(p) = e^{-p\beta} \varphi(cp)^\gamma , \quad p \geq 0 \tag{44}$$

for some $c > 0$, $\beta \in \mathbf{R}$, $\gamma > 1$. This notion will be extended to the whole real line, while rather requiring the FT to satisfy a functional equation of the form

$$\phi(\lambda) = e^{i\lambda\beta} \phi(c\lambda)^\gamma , \quad \lambda \in \mathbf{R} \tag{45}$$

defining two-sided *semistable* laws. When $\beta = 0$, we shall speak of a strictly semistable law.

This notion of *semistability* for sums was first introduced by *Paul Lévy* in 1937[12] (see Lukacs (1983),[11] p. 45 for a survey on this point) and worked out in Lamperti (1962)[13] the context of stochastic processes. Additional references on these topics are Maejima et al. (2000)[14], Maejima and Sato (1999)[15], and Maejima (2001)[16]. The book by Sato (1999)[6] is a most instructive survey.

3.1 One-Sided Semistability

We start with one-sided *semistability*. This notion is central in the construction of all *semistable* distributions. We shall first give the *LST* representation of one-sided *semistable* laws. We shall then enumerate some of their remarkable properties.

3.1.1 *One-sided strictly semistable laws*

Let $\gamma > 1$ and $c \in (0, 1)$. First, consider the class of positive random variables X^+ whose *LST* satisfies the simpler functional equation (of the form (44), with $\beta = 0$)

$$\varphi_{X^+}(p) = \varphi_{X^+}(cp)^{\gamma}, \quad p \geq 0. \tag{46}$$

These variables will be identified with the so-called positive *semistable* variables and may thus be seen as the fixed point of some transformation to be elucidated below as statistical *semi-self-similarity.*

Upon reasoning with the function $L(q) := -\log \varphi_X(e^q)$, this functional equation takes the simpler form $L(q) = \gamma L(q + \log c)$; the class of solutions of (46) are then easily found to be, formally

$$\varphi_{X^+}(p) = \exp -p^{\alpha} e^{\alpha\zeta(\log p)} \tag{47}$$

where $\alpha > 0$ is defined through

$$\gamma c^{\alpha} = 1 \tag{48}$$

and where $\zeta(q)$, $q := \log p$, $p > 0$, is a periodic (with period $\log c$) function on the real line.

As φ_{X^+} must be the *LST* of some positive random variable X^+ additional conditions have to be added. They are

(i) The constant α necessarily belongs to the interval $(0, 1)$. Indeed, if this were not the case, i.e. assuming $\alpha > 1$, assuming $\zeta(q)$ to be periodic and constant would give $\varphi_{X^+}(p) = \exp -sp^{\alpha}$, with $s > 0$. In the case $\alpha > 1$, such $\varphi_{X^+}(p)$ is not completely monotone (i.e. such that for all $p > 0$, $(-1)^n \varphi_{X^+}^{(n)}(p) \geq 0$, $n \geq 0$)[7] and thus cannot be the *LST* of a probability distribution.

(ii) The functional Eq. (46) shows that X^+ is necessarily infinitely divisible. Consequently, the function $I_{X^+}(p) := -\frac{d}{dp} \log \varphi_{X^+}(p)$ must itself be completely monotone.[7] Observing that

$$I_{X^+}(p) = \alpha p^{\alpha-1}(1 + \zeta'(\log p))e^{\alpha\zeta(\log p)} \tag{49}$$

where $\alpha p^{\alpha-1}$ is a completely monotone function if $\alpha \in (0, 1)$, and recalling that the product of two completely monotone functions is completely monotone, the condition that

$$(1 + \zeta'(\log p))e^{\alpha\zeta(\log p)} \tag{50}$$

is completely monotone must be fulfilled. Under this form, it may be a complicated task to supply an example of such ζ.

Remark 4. The functional Eq. (46) is also

$$\varphi_{X^+}(p) = \varphi_{X^+}(p/\bar{c})^{\bar{\gamma}} \tag{51}$$

with $\bar{c} := 1/c > 1$ and $\bar{\gamma} := 1/\gamma \in (0, 1)$ so that the function ζ, which depends on the constant c, must be invariant under the transformation $c \to 1/c$, observing that $\bar{\alpha} := \log \bar{\gamma} / -\log \bar{c} = \alpha$.

We now come to an explicit construction of the LST given in (47), exploiting a correspondence with the Lévy spectral function.[28]

From the infinite divisibility property, the *LST* $\varphi_{X^+}(p)$ of X^+ admits the representation

$$\varphi_{X^+}(p) = \exp - \int_0^{+\infty} (1 - e^{-px}) d\pi(x) \tag{52}$$

where $d\pi$ is some *Lévy* measure (i.e. a positive *Radon* measure) induced by its *Lévy* spectral function $\pi(x)$.

The functional Eq. (46) is consistent with the scaling condition (see also Maejima and Samorodnitsky (1999)[17]) on the *Lévy* spectral function, which is

$$\gamma^{-n}\pi(x) = \pi(\gamma^{n/\alpha}x), \quad n \in \mathbf{Z}. \tag{53}$$

The solutions to this problem are explicitly

$$\pi(x) = -x^{-\alpha} e^{\alpha\nu(\log x)}, x \in (0, +\infty), \quad \alpha \in (0, 1) \tag{54}$$

extending (2). Here, in (54), $\nu(z)$, $z := \log x$, $x > 0$ is a bounded periodic (with period $-\log c$)

function, such that $z - \nu(z)$ is non-decreasing so as to guarantee the monotony of $\pi(x)$.

Example 1. A most simple and fundamental example of such ν is, with $\rho \in (0, 1]$

$$\nu(z) = \frac{\rho \log c}{2\pi} \sin \frac{2\pi z}{-\log c} \tag{55}$$

assuming a single term in the full *Fourier* series expansion of $\nu(z)$.

From this correspondence, the *Fourier* series expansion of $\exp \alpha\zeta(\log p)$ in (47) can easily be obtained from the one of $\exp \alpha\nu(\log x)$ which appears in the *Lévy* spectral function (54), in the following way.

With $\kappa_0 \in \mathbf{R}^+$, $\kappa_n \in \mathbf{C}$ such that, for $n \geq 1$, $\kappa_{-n} = \overline{\kappa}_n$ (the complex conjugate), let

$$\exp \alpha\nu(\log x) := \sum_{n \in \mathbf{Z}} \kappa_n e^{-2i\pi n \log x / \log c} \tag{56}$$

be the *Fourier* series expansion of $\exp \alpha\nu(\log x)$. Note that

$$\kappa_n := \int_{\log c}^{-\log c} e^{\alpha\nu(z)} e^{2i\pi n z} dz\,, \quad n \in \mathbf{Z}$$

are the *Fourier* coefficients.

Combining (54) and (56), we obtain

$$\pi(x) = -\sum_{n \in \mathbf{Z}} \kappa_n x^{-\alpha_n} := \sum_{n \in \mathbf{Z}} \pi_n(x) \tag{57}$$

with

$$\alpha_n = \alpha + in\alpha_c\,, \quad n \in \mathbf{Z}\,, \quad \alpha_c := 2\pi/\log c \tag{58}$$

a complex sequence of tail exponents.

Thus, from (2) and (3)

$$\int_0^{+\infty} (1 - e^{-px}) d\pi(x) = \sum_{n \in \mathbf{Z}} \int_0^{+\infty} (1 - e^{-px}) d\pi_n(x)$$

$$:= \sum_{n \in \mathbf{Z}} s_n p^{\alpha_n} \tag{59}$$

with

$$s_n := \kappa_n \Gamma(1 - \alpha_n)\,, \quad n \in \mathbf{Z} \tag{60}$$

using the complex version of the *Euler* function (Ref. 10, Chap. 7).

From (47) and (52), we therefore identify the *Fourier* series expansion of $e^{\alpha\zeta(\log p)}$ as

$$e^{\alpha\zeta(\log p)} = \sum_{n \in \mathbf{Z}} s_n e^{in\alpha_c \log p} \tag{61}$$

where $(s_n,\, n \in \mathbf{Z})$ is given by (60) in terms of $(\kappa_n\; n \in \mathbf{Z})$.

3.1.2 *Conclusion*

To summarize, a strictly *semistable* model for $X^+ > 0$ requires that

$$\varphi_{X^+}(p) = \exp -s(p) p^\alpha \tag{62}$$

where the log-scale parameter is allowed to vary periodically with $\log p$ according to

$$\log s(p) = \alpha\zeta(\log p)\,. \tag{63}$$

In a *semistable* model, the scale parameter varies in a log-periodic fashion. The function ζ is known as soon as the *Fourier* series expansion of the *Lévy* spectral function for jumps is known.

Let us stress some additional properties of *semistable* positive variables.

3.1.3 *Additional properties of one-sided strictly semistable laws*

We enumerate some of their remarkable properties:

(1) Semi-self-similarity. We start with a notion of statistical *semi-self-similarity.* Let $X^+ := X^+(1)$ be a positive random variable with *LST* φ_{X^+}. With $\varsigma > 0$, define the variable $X^+(\varsigma)$ as the variable whose *LST* is $\varphi_{X^+(\varsigma)} = \varphi_{X^+}^\varsigma$, raising φ_{X^+} to the power ς (in our univariate context, $\varphi_{X^+}^\varsigma$ is the *LST* of some random variable if and only if X^+ is *infinitely divisible*). The variable X^+ will be said to be statistically *self-similar*, with exponent H, if the following holds, for any $\varsigma > 0$

$$X^+(\varsigma) \stackrel{d}{=} \varsigma^H X^+(1)\,. \tag{64}$$

Now, the functional Eq. (46) means that

$$X^+(\gamma) \stackrel{d}{=} c^{-1} X^+\,. \tag{65}$$

With $\alpha > 0$ defined through (48), this is also

$$X^+(\gamma^n) \stackrel{d}{=} \gamma^{n/\alpha} X^+\,, \quad n \in \mathbf{Z}\,. \tag{66}$$

Thus, the class of all such X^+ identifies with a class of statistically *semi-self-similar* positive random variables, with exponent $H = 1/\alpha > 0$, i.e. those for which (64) holds but *only* at points $\varsigma = \gamma^n$, $n \in \mathbf{Z}$. By its construction, a positive *semistable* variable is in the *infinitely divisible* class.

(2) Physical relevance of semi-self-similarity. The physical reason why one should be interested in such *semi-self-similar* variables proceeds as follows: as X^+ is a *infinitely divisible* variable, this basically means that, "roughly" speaking

$$X^+ \overset{d}{=} \sum_{m=1}^{P(1)} Z_m \tag{67}$$

where $P(1)$ is a *Poisson* variable with mean value 1. Here $(Z_m)_{m \geq 1}$ is an *iid* sequence of "local" events. The observed "global" random event X^+ is thus assumed to be the sum of a *Poissonian* number of "local" events which sounds reasonable, physically: the observed variable is a clustering of "micro-events".

What the functional Eq. (46) tells us, in addition, is that this global event X^+ could as well result from *more* local events (replacing $P(1)$ by $P(\gamma)$, $\gamma > 1$, in (67)) but with *smaller* reduced amplitudes substituting cZ_m to Z_m, $m \geq 1$, in (67), in such a way that X^+ is also, for some judicious constants c and γ

$$X^+ \overset{d}{=} \sum_{m=1}^{P(\gamma)} cZ_m\,. \tag{68}$$

It is some sort of an ignorance principle on the way the final global result may be produced: the exact scale and intensity of the observable are undetermined. As a result, there are now *two* basic unknown structure parameters, namely c and γ (or alternatively c and $\alpha = \log \gamma / -\log c$) to deal with.

This illustrates the commonly accepted fact that physical variables of interest are "invariants" in some statistical sense which is made precise here.

We finally underline the analogy of the problem treated in this monograph on a statistical basis and the one of Sornette (1998)[18] on discrete scale invariance arising from renormalization group theory in physics. There, the exponent α is naturally interpreted as a "dimension" and the log-periodic decorations lead to the richer notion of a complex dimension, whereas, in the statistical language advocated here, α simply is a tail exponent and the log-periodic decorations states that the underlying stable variable possibly exhibits an oscillating scale parameter.

(3) Stable laws are semistable. Note that a constant function $\zeta(q) := \zeta$ in (47) satisfies all the requirements, so that a positive *stable* model is a particular case of a *semistable* model (see Remark 4 of Sec. 2.3). *Semistability* identifies here with the notion of *semi-self-similarity*, which is weaker than *stability*, leading to a larger class of distributions. Although this assertion is true for one-sided strictly semistable, it turns out to be a general fact.

(4) Semi-self-decomposability. One-sided strictly semistable laws with support $[0, \infty)$ are thus infinitely divisible laws; they contain the class of stable laws. Actually, semistable distributions constitute a subclass of semi-self-decomposable (infinitely divisible) laws. To see this, let us recall that a positive random variable X^+ is said to be semi-self-decomposable (Ref. 6, p. 90) if, for some $c \in (0, 1)$

$$X^+ \overset{d}{=} cX^+ + R$$

where R is an *ID* positive random variable, independent of cX^+ (note that a stable distribution is self-decomposable in that the above identity in law holds for *any* such c, with $R \overset{d}{=} (1 - c^\alpha)^{1/\alpha} X^+$). In terms of *LST*, this means that $\varphi_{X^+}(p)$ may be decomposed into

$$\varphi_{X^+}(p) = \varphi_{X^+}(cp)\varphi_R(p)\,, \quad p \geq 0$$

where $\varphi_R(p)$ is the *LST* of the distribution of R. Now, from (46)

$$\varphi_{X^+}(p) = \varphi_{X^+}(cp)\varphi_{X^+}(cp)^{\gamma-1}\,, \quad p \geq 0\,.$$

Thus

$$\varphi_R(p) = \varphi_{X^+}(cp)^{\gamma-1}$$

which is the *LST* of a probability distribution because X^+ is *ID* and $\gamma > 1$. Positive self-decomposable (stable) distributions derive their importance from the fact they are the limit laws for sums of independent (respectively identically distributed) random variables after a convenient scaling increasing with the size of the sample (see Loève (1977),[19] Petrov (1975),[20] and van Harn et al. (1982),[21] pp. 82–85).

(5) Heavy-tailedness. It may be shown that under some additional regularity property of function ν that

$$\Pr(X^+ > x) \sim x^{-\alpha} e^{\alpha\nu(\log x)} \tag{69}$$

for large x. Thus X^+ is close to be heavy-tailed (in the standard sense of regular variation) with tail index $\alpha > 0$, just like the *positive stable* model is. More precisely, assuming ν to be *Lipschitz* (as ν is periodic this condition is fulfilled if ν is of class C^1), one may easily check that the cpdf $\bar{G}(x) = \frac{\pi(x)}{\pi(1)}$, $x > 1$ is subexponential so that from Embrechts et al. (1997),[22] p. 581, Theorem A3.22, $\Pr(X^+ > x) \sim -\pi(x)$ which is (69). Indeed, if this is so, with $\overline{G^{*2}}(x) := \bar{G}(x) + \int_1^x \bar{G}(x-z)dG(z)$, one must check $\lim_{x\to\infty} \overline{G^{*2}}(x)/\bar{G}(x) = 2$ or equivalently that

$$\lim_{x\to\infty} \int_1^x \left(1-\frac{z}{x}\right)^{-\alpha} e^{\alpha[\nu(\log(x-z))-\nu(\log(x))]} dG(z) = 1$$

which is true if ν is *Lipschitz.*

(6) Empirical evidence of semistability. This goes through the observation that

$$-\log(-\log \varphi_{n,X^+}(p)) = \alpha(\log p - \zeta(\log p)) \quad (70)$$

i.e. that a plot of $-\log(-\log \varphi_{n,X^+}(p))$ against $\log p$ should exhibit oscillations around a linear trend with positive slope α. Here, φ_{n,X^+} is the *empirical LST* that can be obtained from an *n-iid* sample, say $(x_m^+,\ m \geq 1)$ of X^+ as

$$\varphi_{n,\,X^+}(p) = \frac{1}{n}\sum_{m=1}^{n} e^{-px_m^+}\,. \quad (71)$$

From this remark, it follows that fitting a semistable model to data requires to find the theoretical *LST* of the form (47) which minimizes some functional distance to the empirical *LST* (71).

Note also that the theoretical *LST* of X^+ is such that, with $k \in \mathbf{Z}$, $\varphi_{X^+}(c^{-k}) = \exp -sc^{\alpha k}$ with $\log s = \alpha\zeta(0)$: evaluating φ_{X^+} at geometrically scattered points c^{-k}, $k \in \mathbf{Z}$, the oscillating part vanishes.

3.1.4 *Shifted one-sided semi-stable laws*

Actually, the variable X^+ does not cover all the class of one-sided *semistable* variables. Those obtained after a shift of X^+ are also in this class as we now show.

Let $x^+ \in \mathbf{R}$. Consider the shifted variable $\tilde{X}^+ := X^+ + x^+$. The shifted variable now satisfies the functional equation of the type (44), with $\beta = x^+(1-\gamma c)$

$$\varphi_{\tilde{X}^+}(p) = e^{-p\beta}\varphi_{\tilde{X}^+}(cp)^\gamma\,, \quad p \geq 0 \quad (72)$$

whose solution is

$$\varphi_{\tilde{X}^+}(p) = \exp -[px^+ + p^\alpha e^{\alpha\zeta(\log p)}]\,, \quad p \geq 0\,. \quad (73)$$

This accounts for one-sided *semistable* model, with support unbounded to the right. Note that the random variable $\tilde{X}^+$ is a shifted one-sided semistable variable with support unbounded to the left. In this formulation, the observed random event $\tilde{X}^+$ still is the sum of a *Poissonian* number of "micro-events," as in (67). However, from the functional Eq. (72), this global event $\tilde{X}^+$ could as well result from more local events but with reduced *and* shifted amplitudes, in such a way that $\tilde{X}^+$ is also, for some judicious constants c, γ and β

$$\tilde{X}^+ = \sum_{m=1}^{P(\gamma)} c(Z_m - \beta)\,. \quad (74)$$

The exact way the final global result may be produced is unknown in that the exact scale, intensity *and* location of the observable are now basically undetermined. Thus *semistability* extends the notion of *semi-self-similarity* for positive variables discussed above in that the location parameter of the observable is also unknown: in this sense, it is a "broad sense" *semi-self-similarity.*

3.2 Two-Sided Semistability

As for the case of stable random variables, it is possible to design a two-sided version of one-sided semistable random variables. We shall again distinguish three cases.

3.2.1 *Exponent* $\alpha \in (0, 1)$

The general *FT* representation for two-sided strictly semistable laws.

Consider two independent one-sided *Lévy* variables, say X_1^+ and X_2^+, with respective *Lévy* spectral function for positive jumps $x > 0$

$$\pi_l(x) = -x^{-\alpha}e^{\alpha\nu_l(\log x)}\,, \quad l = 1, 2 \quad (75)$$

where ν_l, $l = 1, 2$ are periodic functions with the *same* period, $-\log c$.

Define next the full *Lévy* spectral function for jumps on the interval $(-\infty, 0) \cup (0, +\infty)$, as

$$\pi(x) = \pi_1(x)\mathbf{1}\ (x > 0) - \pi_2(-x)\mathbf{1}\ (x < 0)\,. \quad (76)$$

Next consider the real-valued random variable

$$X = X_1^+ - X_2^+\,. \quad (77)$$

It has *Fourier* transform (FT)

$$\phi_X(\lambda) := \mathbf{E}e^{i\lambda X} = \varphi_{X_1^+}(-i\lambda)\varphi_{X_2^+}(i\lambda) \tag{78}$$

which satisfies (45) with $\beta = 0$.

In more explicit form

$$\phi_X(\lambda) = \exp - \int_{-\infty}^{+\infty} (1 - e^{i\lambda x}) d\pi(x)$$
$$= \exp -[(-i\lambda)^\alpha e^{\alpha\zeta_1(\log -i\lambda)} + (i\lambda)^\alpha e^{\alpha\zeta_2(\log i\lambda)}] . \tag{79}$$

For $l = 1, 2$, let

$$e^{\alpha\zeta_l(q)} = \sum_{n\in\mathbf{Z}} s_{n,l} \exp in\alpha_c q \tag{80}$$

stand for the *Fourier* series expansion of the real-valued periodic functions $e^{\alpha\zeta_l(q)}$, $l = 1, 2$. Here, the complex *Fourier* coefficients are such that $s_{0,l} \in \mathbf{R}^+$, $(s_{-n,l} = \bar{s}_{n,l})_{n\geq 1}$, $l = 1, 2$. We shall represent these numbers by $s_{n,l} := |s_{n,l}|e^{i\varpi_{n,l}}$. With $(\alpha_n,\, n \in \mathbf{Z})$ defined in (58), we obtain

$$\phi_X(\lambda) = \exp - \sum_{n\in\mathbf{Z}} (s_{n,1}(-i\lambda)^{\alpha_n} + s_{n,2}(i\lambda)^{\alpha_n}) . \tag{81}$$

This expression can be re-arranged to give

$$\phi_X(\lambda) = \exp - \sum_{n\in\mathbf{Z}} s_n(\lambda)|\lambda|^\alpha (1 + i\rho_n(\lambda)) \tag{82}$$

with

$$s_n(\lambda) = |s_{n,1}|e^{n\frac{\pi}{2}\alpha_c \operatorname{sign}(\lambda)} \cos \psi_{n,1} + |s_{n,2}|e^{-n\frac{\pi}{2}\alpha_c \operatorname{sign}(\lambda)} \cos \psi_{n,2} \tag{83}$$

and

$$\rho_n(\lambda) := \frac{1}{s_n(\lambda)} [|s_{n,1}|e^{n\frac{\pi}{2}\alpha_c \operatorname{sign}(\lambda)} \sin \psi_{n,1} + |s_{n,2}|e^{-n\frac{\pi}{2}\alpha_c \operatorname{sign}(\lambda)} \sin \psi_{n,2}] . \tag{84}$$

In (83) and (84), the "phase" constants which appear are given by

$$\begin{aligned} \psi_{n,1} &:= n\alpha_c \log|\lambda| - \frac{\pi}{2}\alpha \operatorname{sign}(\lambda) + \varpi_{n,1} \\ \psi_{n,2} &:= n\alpha_c \log|\lambda| + \frac{\pi}{2}\alpha \operatorname{sign}(\lambda) + \varpi_{n,2} . \end{aligned} \tag{85}$$

The FT for two-sided strictly semistable random variables defined in (82) extends the one in (17) for two-sided strictly stable random variables. Here, we get an enumerable set of scale and skewness parameters $(s_n(\lambda),\, \rho_n(\lambda))_{n\geq 1}$, defined by (83), (84) and (85) all varying with λ.

The *FT* representation for *symmetric* strictly two-sided semistable laws.

We now exhibit the form of the *symmetric* semistable characteristic function, assuming $\zeta_1(q) = \zeta_2(q)$. To do this, we shall rather work with the following alternative *Fourier* series expansion of the periodic functions $e^{\alpha\zeta(q)} := e^{\alpha\zeta_1(q)} = e^{\alpha\zeta_2(q)}$:

$$e^{\alpha\zeta(q)} = s + \sum_{n\geq 1} a_n \cos n\alpha_c q + \sum_{n\geq 1} b_n \sin n\alpha_c q . \tag{86}$$

Here, the coefficients $s > 0$, $(a_n,\, b_n)_{n\geq 1}$ are all real, i.e. $a_n := s_n + \bar{s}_n$ and $b_n := i(s_n - \bar{s}_n)$, with $s := s_{0,1} = s_{0,2} = \Gamma(1-\alpha)\kappa > 0$, $s_n := s_{n,1} = s_{n,2}$, $n \geq 1$.

Under this hypothesis, the characteristic function (82) takes the simpler real-valued form

$$\phi_X(\lambda) = \exp -\tilde{s}(|\lambda|)|\lambda|^\alpha \tag{87}$$

where $\tilde{s}(|\lambda|) := e^{\alpha\tilde{\zeta}(\log|\lambda|)}$ is a unique periodic scale parameter. Here the function $e^{\alpha\tilde{\zeta}(q)}$ admits a *Fourier* series expansion of a form similar to the one in (86). It is

$$e^{\alpha\tilde{\zeta}(q)} := \tilde{s} + \sum_{n\geq 1} \tilde{a}_n \cos n\alpha_c q + \sum_{n\geq 1} \tilde{b}_n \sin n\alpha_c q \tag{88}$$

where

$$\tilde{s} = 2\cos\frac{\pi\alpha}{2} s = 2\cos\frac{\pi\alpha}{2}\Gamma(1-\alpha)\kappa \tag{89}$$

and

$$\begin{aligned} \tilde{a}_n &= 2\cos\frac{\pi\alpha}{2}\cosh\frac{\pi^2 n}{\log c} a_n - 2\sin\frac{\pi\alpha}{2}\sinh\frac{\pi^2 n}{\log c} b_n \\ \tilde{b}_n &= 2\sin\frac{\pi\alpha}{2}\sinh\frac{\pi^2 n}{\log c} a_n + 2\cos\frac{\pi\alpha}{2}\cosh\frac{\pi^2 n}{\log c} b_n \end{aligned} \tag{90}$$

are the new *Fourier* coefficients expressed in terms of the ones s, $(a_n,\, b_n)_{n\geq 1}$ of the original function scale function $\exp \alpha\zeta(q)$.

A symmetric semistable random variable admits the FT (82) with all skewness coefficients null $\rho_n(\lambda) = 0$, $n \geq 0$ and with unique scale parameter

$$\tilde{s}(|\lambda|) := e^{\alpha\tilde{\zeta}(\log|\lambda|)} = \sum_{n\in\mathbf{Z}} s_n(\lambda)$$

defined in (88), (89) and (90).

The general *FT* representation for shifted two-sided semistable laws.

Let $x \in \mathbf{R}$. Consider the shifted variable $\tilde{X} := X + x$. From (82), the shifted variable now admits the *FT*

$$\phi_{\tilde{X}}(\lambda) = \exp\, i\lambda x - \sum_{n\in\mathbf{Z}} s_n(\lambda)|\lambda|^\alpha(1+i\rho_n(\lambda)) \quad (91)$$

with a shift at point x.

From the hypothesis $\alpha \in (0, 1)$, the number $x_1 := \int_{|x|\leq 1} x d\pi(x)$ is finite, so that, from (79)

$$\phi_X(\lambda) = \exp\, i\lambda x_1 - \left[\int_{-\infty}^{+\infty} (1+i\lambda x\mathbf{1}\ (|x|\leq 1) - e^{i\lambda x})d\pi(x)\right]. \quad (92)$$

Thus, with $\tilde{x} := x + x_1$, the *FT* of $\tilde{X}$ also admits the representation

$$\phi_{\tilde{X}}(\lambda) = \exp\, i\lambda\tilde{x} - \left[\int_{-\infty}^{+\infty} (1+i\lambda x\mathbf{1}\ (|x|\leq 1) - e^{i\lambda x})d\pi(x)\right] \quad (93)$$

which is also

$$\phi_{\tilde{X}}(\lambda) = \exp\, i\lambda(\tilde{x}-x_1) - \sum_{n\in\mathbf{Z}} s_n(\lambda)|\lambda|^\alpha(1+i\rho_n(\lambda)). \quad (94)$$

By (92), (93) and (94), we note that the *Lévy-Khintchine* representation of strictly semistable variables on the real line are obtained while taking $\tilde{x} = x_1$.

3.2.2 *Exponent* $\alpha \in (1, 2)$

Just like for stable laws, the above construction, valid for $\alpha \in (0, 1)$ can be extended to the range $\alpha \in (1, 2)$.

The general *FT* representation for strictly and shifted two-sided semistable laws.

Proceeding as for stable random variables, with $p \geq 0$ and using (57), (59) and (75), we shall rather use the identity for $l = 1, 2$

$$\int_0^{+\infty} (1 - px - e^{-px})d\pi_l(x) = p^\alpha \sum_{n\in\mathbf{Z}} \kappa_{n,l}\Gamma(1-\alpha_n)p^{in\alpha_c} = p^\alpha e^{\alpha\zeta_l(\log p)} \quad (95)$$

extending (24).

As a result,

$$\int_{-\infty}^{+\infty} (1 + i\lambda - e^{i\lambda x})d\pi(x) = (-i\lambda)^\alpha e^{\alpha\zeta_1(\log -i\lambda)} + (i\lambda)^\alpha e^{\alpha\zeta_2(\log i\lambda)} = \sum_{n\in\mathbf{Z}} s_n(\lambda)|\lambda|^\alpha(1+i\rho_n(\lambda)). \quad (96)$$

Note that in this parameter range, $x_1 := -\int_{|x|>1} \cdot x d\pi(x) < \infty$. Hence,

$$\int_{-\infty}^{+\infty} (1 + i\lambda x\mathbf{1}\ (|x|\leq 1) - e^{i\lambda x})d\pi(x) = \int_{-\infty}^{+\infty} (1 + i\lambda x - e^{i\lambda x})d\pi(x) + i\lambda x_1. \quad (97)$$

Let $\tilde{x} \in \mathbf{R}$. Define the *FT* of a shifted semistable *Lévy* variable, as in the case $\alpha \in (0, 1)$, by

$$\phi_{\tilde{X}}(\lambda) = \exp\, i\lambda\tilde{x} - \int_{-\infty}^{+\infty} (1 + i\lambda x\mathbf{1}\ (|x|\leq 1) - e^{i\lambda x})d\pi(x). \quad (98)$$

From (96) and (97), we get

$$\phi_{\tilde{X}}(\lambda) = \exp\, i\lambda(\tilde{x}-x_1) - \sum_{n\in\mathbf{Z}} s_n(\lambda)|\lambda|^\alpha(1+i\rho_n(\lambda)). \quad (99)$$

From the *Lévy-Khintchine* representation, taking $\tilde{x} - x_1 = 0$ yields the *FT* of a strictly semistable *Lévy* stable variable on the real line, say X. It is

$$\phi_X(\lambda) = \exp - \sum_{n\in\mathbf{Z}} s_n(\lambda)|\lambda|^\alpha(1+i\rho_n(\lambda)). \quad (100)$$

3.2.3 *Exponent* $\alpha = 1$

For this value of the exponent, special care is needed.

The general *FT* representation for strictly and shifted two-sided semistable laws.

In this critical case, with $p \geq 0$, the following identity, extending (31), is of some use

$$\int_0^{+\infty} (1 - p \sin x - e^{-px}) d\pi_l(x) = -p \log p e^{\nu_l(\log p)}, \quad l = 1, 2. \quad (101)$$

where

$$\pi_l(x) = -x^{-1} \sum_{n \in \mathbf{Z}} \kappa_{n,l} x^{-in\alpha_c} := -x^{-1} e^{\nu_l(\log x)}. \quad (102)$$

As a result, with π the full *Lévy* measure for jumps defined by (76) and (102),

$$\int_{-\infty}^{+\infty} (1 + i\lambda \sin x - e^{i\lambda x}) d\pi(x) = i\lambda \log(-i\lambda) e^{\nu_1(\log -i\lambda)} - i\lambda \log(i\lambda) e^{\nu_2(\log i\lambda)}. \quad (103)$$

Observing that, with $x_1 := \int_{-\infty}^{+\infty} (x\mathbf{1}\ (|x| \leq 1) - \sin x) d\pi(x) < \infty$, we have

$$\int_{-\infty}^{+\infty} (1 + i\lambda \sin x - e^{i\lambda x}) d\pi(x) = \int_{-\infty}^{+\infty} (1 + i\lambda x \mathbf{1}\ (|x| \leq 1) - e^{i\lambda x}) d\pi(x) - i\lambda x_1. \quad (104)$$

We finally get the *FT* for the shifted *Lévy* variable in the case $\alpha = 1$

$$\phi_{\tilde{X}}(\lambda) = \exp i\lambda(\tilde{x} - x_1) - [i\lambda \log(-i\lambda) e^{\nu_1(\log -i\lambda)} - i\lambda \log(i\lambda) e^{\nu_2(\log i\lambda)}]. \quad (105)$$

Assuming $\nu_1 = \nu_2 = \nu$, we get the form of the characteristic function for strictly 1-*semistable* variables

$$\phi_{\tilde{X}}(\lambda) = \exp i\lambda(\tilde{x} - x_1) - [i\lambda \log(-i\lambda) e^{\nu(\log -i\lambda)} - i\lambda \log(i\lambda) e^{\nu(\log i\lambda)}]. \quad (106)$$

3.2.4 *Concluding remarks*

Random variables whose *Fourier* transforms are given by (82), (100) and (106) are known as two-sided strictly semistable *Lévy* variables.

The *FT* representation for symmetric semistable variables is given in (87)–(90). A remarkable feature of symmetric *semistable* variables is that they may be constructed from the subordination of a symmetric *stable* variable, with a positive *semistable* "subordinator," as will be shown in Subsec. 3.4.

Random variables whose *Fourier* transforms are given by (94), (99) and (105) are known as shifted two-sided semistable *Lévy* variables. They constitute the solutions to the functional Eq. (45).

Strict or shifted two-sided semistable *Lévy* variables share properties very similar to the ones enumerated in Subsec. 3.1.3 for their one-sided version.

Before proceeding, let us supply an additional remark, underlining the importance of *semistable* models in statistics.

3.3 Semi-stable Models as Limit Laws in Statistics

One-sided *semistable* variables have been shown to be semi-self-decomposable. It can easily be shown that this is also true of two-sided *semistable* variables. Semi-self-decomposable distributions derive their importance from the fact they are the limit laws for sums of independent random variables after a suitable location-scale transform. It turns out that, owing to their *semi-self-similarity*, *semistable* variables are also the limit laws for sums of normalized independent *and* identically distributed random variables when data are aggregated in geometrical packets.

Let X be any *semistable* variables with *LST* or *FT* either given by (47), (73), (82), (00), (94), (99), (100) and (105). These variables also appear as the only possible non-degenerate limit laws in the *statistics for sums* problem in the following wider (geometrical) sense: if X is *semistable*, then, as it can easily be checked, there exists a random variable $\mathcal{X}$ and three sequences $x_n \in \mathbf{R}$, $\sigma_n > 0$ and $\gamma_n > 0$, such that

$$\sum_{m=1}^{\gamma_n} \frac{\mathcal{X}_m - x_n}{\sigma_n} \xrightarrow{d} X \text{ as } n \uparrow +\infty \quad (107)$$

where $\mathcal{X}_m \stackrel{d}{=} \mathcal{X}$, $m \geq 1$ is an independent sequence. The integer-valued sequence γ_n is assumed to satisfy the additional geometrical growth properties: $\lim_{n\uparrow+\infty} \gamma_n = +\infty$ and $\lim_{n\uparrow\infty} \gamma_{n+1}/\gamma_n = \gamma \geq 1$.

The variable $\mathcal{X}$ is said to belong to the domain of attraction (DA) of X (note that again X itself belongs to its own DA).[2–5]

Thus, *semistable* distributions derive their importance in statistics from the fact that they are the limit laws of the sum of a *geometric* number γ_n of *iid* random variables $\mathcal{X}_m$, $m \geq 1$, after a convenient location-scale transform. *Semistable* variables also form a proper subclass of ID and semi-self-decomposable variables. *Stable* variables are *semistable* and can be obtained in the limit $\gamma \downarrow 1$, $c \uparrow 1$, while $\frac{\log \gamma}{-\log c} = \alpha$. This latter class is therefore an extension of the former one, hence its name.

3.4 Symmetric Semi-stable Variables as Subordinates

Let $\alpha_1 \in (0, 1)$ and X^+ be a one-sided *Lévy*-semistable variable with index α_1 hence with *LST*

$$\varphi_{X^+}(p) := \mathbf{E}e^{-pX^+} = \exp -p^{\alpha_1} e^{\alpha_1 \zeta(\log p)}$$

given by (47), substituting α_1 to α. Here, the function $\zeta(q)$ is periodic with period $\log c_1$, with $\gamma c_1^{\alpha_1} = 1$.

Let $(S(t), t \geq 0)$, $S(0) = 0$, be a symmetric *stable* motion with parameter $\alpha \in (0, 2)$. As a result, from (36) and (37), with $s = 2\Gamma(1-\alpha) \cos \frac{\pi\alpha}{2} \kappa$

$$\mathbf{E}e^{i\lambda S(t)} = \exp -ts|\lambda|^{\alpha} .$$

Assume X^+ and $(S(t), t \geq 0)$ are independent and consider the random variable

$$X = S(X^+) .$$

Conditioning, we get, with $\alpha_2 := \alpha_1 \alpha \in (0, 2)$

$$\begin{aligned} \phi_X(\lambda) &:= \mathbf{E}e^{i\lambda X} \\ &= \mathbf{E} \exp -s|\lambda|^{\alpha} X^+ \\ &= \exp -s^{\alpha_1} e^{\alpha_1 \zeta(\log s + \alpha \log|\lambda|)} |\lambda|^{\alpha_2} \end{aligned} \quad (108)$$

which, from (87), is the *FT* of a symmetric two-sided α_2-semistable *Lévy* variable with scale parameter $\tilde{s}(|\lambda|) := e^{\alpha_2 \tilde{\zeta}(\log|\lambda|)} = s^{\alpha_2/\alpha} \cdot e^{\alpha_2/\alpha \zeta(\log s + \alpha \log|\lambda|)}$, identifying the function $\tilde{\zeta}(\log|\lambda|)$. We note that $\tilde{\zeta}(q)$ is periodic with period $\log c_2$, with $c_2 = c_1^{1/\alpha}$, for which $\gamma c_2^{\alpha_2} = 1$, as required.

4. THE LOG-SEMISTABLE MODELS

It is a very common feature in the natural sciences that *observable* are modeled through the logarithms of other quantities with much more "physical" meaning, such as "energy".[23,24] The distinctive feature of the logarithmic scale is that it measures the distance between two values through their ratio rather than their difference, which amounts, when these values are close, to work with their *relative* (rather than absolute) variation; this transformation thus supplies a discrimination power between two signals which is insensitive to their absolute intensities, rather dealing with the ratio of these intensities. Thus, the intensity of noise, as perceived by the human ear, is usually measured in decibels, i.e. using a logarithmic scale. Similarly, earthquake magnitudes are determined from the logarithm of the amplitudes of waves recorded by seismographs.

In our context, this means that *semistable* variables are to be considered as the observable of some physical phenomenon, say Z, which proves itself *log-semistable.* We shall distinguish between one-sided *log-semistable* and two-sided *log-semistable* variables.

4.1 One-sided Log-Semi-Stability

We shall thus be interested in the model

$$Z = \exp \tilde{X}^+ > z_+ := e^{x^+} > 0 \quad (109)$$

where $\tilde{X}^+$ is the general one-sided *semistable* variable with *LST* defined in (73) with ν *Lipshitz.*

As a result, (109) now defines a variate which is *log-semistable* distributed in the sense that $\log Z$ simply is *semistable* distributed. This model will be shown to present a certain number of interesting features.

Combining (109) with (73), yields the moment generating function of the one-sided *log-semistable* variable. It is

$$\mathbf{E}Z^{-p} = z_+^{-p} \exp -p^{\alpha} e^{\alpha \zeta(\log p)} , \quad p \geq 0 . \quad (110)$$

From this expression, it is clear that Z only possesses *negative* moments. Alternatively, $U := 1/Z$ has all positive moments finite. In addition, we note that $\mathbf{E}Z^{-p} = \mathbf{E}U^p$ is the *Mellin* transform of the distribution F_U of $U := 1/Z$

$$\mathbf{E}U^p = \int_0^{+\infty} u^p F_U(du) . \quad (111)$$

Let us here stress the following additional point. Combining (109) with (69), we get the tail equivalence

$$\Pr(Z > z) \underset{z \uparrow +\infty}{\sim} \log(z/z_+)^{-\alpha} e^{\alpha\nu(\log\log(z/z_+))}. \tag{112}$$

From this expression, it turns out that the random variable Z has tails heavier than the ones of any power law in the sense that, for any power-law tail index $b > 0$

$$\frac{\Pr(Z > z)}{z^{-b}} \underset{z \uparrow +\infty}{\to} +\infty. \tag{113}$$

Such distributions are said to be with tail index zero or with very heavy tails. As a result, they exhibit no finite moment of any arbitrary positive order: the exponential of $\tilde{X}^+$ has a tail that is much fatter than the ones of $\tilde{X}^+$ itself which are with tail index $\alpha > 0$.

4.2 Two-Sided Log-Semi-Stability

Let Z be any real-valued variable with density $f_Z(z)$. Let

$$\varpi_Z^1(\lambda) := \mathbf{E}|Z|^{i\lambda}, \quad \varpi_Z^2(\lambda) := \mathbf{E}|Z|^{i\lambda}\,\mathrm{sign}\,Z,$$

$$W_Z(\lambda) := \begin{bmatrix} \varpi_Z^1(\lambda) & 0 \\ 0 & \varpi_Z^2(\lambda) \end{bmatrix}. \tag{114}$$

The matrix $W_Z(\lambda)$ is known as the characteristic transform (CT) of the variable Z and was introduced in Zolotarev (1962).[25] Just like the characteristic function, or FT, $\phi_Z(\lambda) := \mathbf{E}e^{i\lambda Z}$, the CT (114) is known to characterize the distribution of Z (see Ref. 1, p. 154). If $Z > 0$, we note that $W_Z(\lambda) := \varpi_Z(\lambda)\mathbf{I}$, where $\varpi_Z(\lambda) := \mathbf{E}Z^{i\lambda}$ and $\mathbf{I}$ is a 2×2 identity matrix: the information is unidimensional in this particular case. We shall next introduce the model

$$Z = \exp \tilde{X} > 0 \tag{115}$$

where $\tilde{X}$ is any general two-sided *semistable Lévy* variable defined in Subsec. 3.2. We shall say that Z is a two-sided log-*semistable* variable, in the standard sense that $\log Z$ is a two-sided *semistable observable.*

The characteristic transform of the variable Z is now obtained easily by combining the FT (94), (99) or (105) with (115). It is, simply

$$\varpi_Z(\lambda) = \mathbf{E}e^{i\lambda\tilde{X}} = \phi_{\tilde{X}}(\lambda) \tag{116}$$

for each value of the parameter $\alpha \in (0, 1)$, $\alpha \in (1, 2)$ or $\alpha = 1$. A two-sided *log-semistable* variable, say Z, is thus a positive random variable whose CT $\varpi_Z(\lambda)$ satisfies a functional equation of the type (45)

$$\varpi_Z(\lambda) = e^{i\lambda\beta}\varpi_Z(c\lambda)^{\gamma} \tag{117}$$

for some $c > 0$, $\beta \in \mathbf{R}$, $\gamma > 1$. Note that Z has no moment at all, either positive or negative, apart of course from a moment of order zero.

When dealing with log-*semistable* variables, working with *Mellin* or *Zolotarev* transforms seems appropriate.

4.3 Log-Semistable Models as Limit Laws Under Power Normalization in Statistics

Let Z be any *log-semistable* variables just defined, either one-sided or two-sided. For each such variable, $Z = \exp X$. Here $X = \tilde{X}^+$, or $X = \tilde{X}$ is a *semistable* random variable for which, following (107), there exists a random variable $\mathcal{X}$ in their DA and three sequences $x_n \in \mathbf{R}$, $\sigma_n > 0$ and a geometrical series $\gamma_n > 0$ such that

$$\sum_{m=1}^{\gamma_n} \frac{\mathcal{X}_m - x_n}{\sigma_n} \stackrel{d}{\to} X \text{ as } n \uparrow +\infty \tag{118}$$

where $\mathcal{X}_m \stackrel{d}{=} \mathcal{X}$, $m \geq 1$, independent. As a result, for any Z log-semistable, there exists a random variable $\mathcal{Z} := e^{\mathcal{X}} > 0$ in their DA and three sequences $z_n > 0$, $\sigma_n > 0$ and $\gamma_n > 0$ such that, with $z_n = \exp x_n$,

$$\prod_{m=1}^{\gamma_n} \left(\frac{\mathcal{Z}_m}{z_n}\right)^{1/\sigma_n} \stackrel{d}{\to} Z \text{ as } n \uparrow +\infty. \tag{119}$$

Thus, *log-semistable* distributions derive their importance from the fact that they are the limit laws of the *product* of a *geometric* number γ_n of *iid* random variables $\mathcal{Z}_m$, $m \geq 1$, after a convenient *power normalization.*

4.4 Large Deviation From the Mean

As was noted above, the energy variable Z is always with tail index zero. In all these situations, the maximum $Z_{n:n} := \max(Z_1, \ldots, Z_n)$ of an n-sample $(Z_1, \ldots, Z_n)$ is tail equivalent to the sum $\bar{Z}_n := \sum_{m=1}^{n} Z_m$, in the sense that[22]

$$\frac{\Pr(Z_{n:n} > x)}{\Pr(\bar{Z}_n > x)} \underset{x \uparrow \infty}{\to} 1 . \tag{120}$$

For subexponential distributions, the tail of the maximum determines the tail of the sum.

This holds true for any subexponential variables. Actually, with distributions with tail index zero, two stronger convergence results actually hold. They are

$$\frac{Z_{n:n}}{\bar{Z}_n} \to 1 \text{ (in probability)} \tag{121}$$

and even almost surely if and only if: $\alpha < 1/2$.[26,27]

In this case, a single event explains (in probability or even almost surely) a cumulative event. The understanding of a cumulative energy goes through the one of its largest term.

When $\alpha \in (1, 2)$, from the law of large numbers, the empirical mean $\frac{1}{n}\bar{X}_n := \frac{1}{n} \sum_{m=1}^{n} X_m$ converges almost surely to the theoretical mean, say $\bar{m}_X := \mathbf{E}X$ of X, known to exist in this parameter range: although of power-law type, the tails of X are light enough to guarantee the existence of a mean value. Large deviation theory is then concerned with the evaluation of the (small) probability

$$\Pr\left(\frac{1}{n}\,\bar{X}_n > x\right) \tag{122}$$

as x exceeds the mean $\bar{m}_X$. More precisely, it is concerned with the rate at which this probability tends to zero, as a function of the sample size n. We shall distinguish between the one-sided and two-sided cases.

4.4.1 *One-sided: $X = \tilde{X}^+$*

The function $\varphi_X(p) := \mathbf{E}e^{-pX}$, $p \geq 0$ is not defined in an open neighborhood of $p = 0$, so that the standard large deviation result does *not* hold in this heavy-tailed case: the number $\Pr(\frac{1}{n}\,\bar{X}_n > x)$ does not tend to zero exponentially fast as n goes to infinity, but slower. Indeed, from the tail equivalence of the maximum and sum for subexponential distributions, one gets for large n

$$\Pr\left(\frac{1}{n}\,\bar{X}_n > x\right) \sim \Pr\left(\frac{1}{n}\,X_{n:n} > x\right) = 1 - \Pr(X \leq nx)^n .$$

From (69), with $x > \bar{m}_X$, this is also

$$1 - (1 - \Pr(X > nx))^n \underset{n \uparrow +\infty}{\sim} n(nx)^{-\alpha} e^{\alpha\nu(\log(nx))}$$

showing that, for some $C > 0$

$$\limsup_{n \uparrow \infty} n^{\alpha - 1} \Pr\left(\frac{1}{n}\,\bar{X}_n > x\right) = Cx^{-\alpha} . \tag{123}$$

This formula exhibits a power-law decay to zero of $\Pr(\frac{1}{n}\,\bar{X}_n > x)$, with exponent $1 - \alpha < 0$. Due to the presence of heavy tails, with index $\alpha > 1$ in X, the number $\Pr(\frac{1}{n}\,\bar{X}_n > x)$ only tends algebraically fast to zero as n goes to infinity.

4.4.2 *Concluding remarks*

Finally, we remark that the arithmetic mean for the observed data $(X_1, \ldots, X_n)$, say $\frac{1}{n} \sum_{m=1}^{n} X_m$, corresponds to a *geometric* mean of the hidden energy records $(Z_1, \ldots, Z_n)$:

$$\prod_{m=1}^{n} Z_m^{1/n} .$$

When $\alpha \in (1, 2)$, the law of large numbers which states that $\frac{1}{n}\,\bar{X}_n \to \bar{m}_X$ almost surely obviously reads in terms of the energy records

$$\prod_{m=1}^{n} Z_m^{1/n} \underset{n \uparrow \infty}{\to} \exp \bar{m}_X , \quad \text{almost surely} . \tag{124}$$

This observation and the large deviation results just mentioned show that, for $\alpha \in (1, 2)$, there are large deviation results for the energy sequence itself $(Z_m, m \geq 1)$, but not in terms of its empirical arithmetic mean (it simply could not converge as a result of very heavy-tailedness of its distribution), rather of its empirical geometrical mean. In explicit form, (123) reads in terms of the geometrical mean of the energies

$$\limsup_{n \uparrow \infty} n^{\alpha - 1} \Pr\left(\prod_{m=1}^{n} Z_m^{1/n} > z\right) = C[\log z]^{-\alpha} . \tag{125}$$

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