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On the physical relevance of max- and log-max-selfsimilar distributions

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Abstract. This work emphasizes the special role played by *max-semistable* and *log-max-semistable* distributions as relevant statistical models of various observable and "internal" variables in Physics. Some of their remarkable properties (chiefly *self-similarity*) are displayed in some detail. One of their characteristic features is a log-periodic variation of the scale parameter which appears in the *stable* extreme value distributions.

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1 Introduction

This work emphasizes the special role played by *max-semistable* and *log-max-semistable* distributions as relevant statistical models of various observable and "internal" variables arising from the Natural Sciences such as, (say), Hydrology, Geophysics, Finance... The fitting of such distributions to real-world problems is not addressed here and is postponed to a future work, rather the chief objective is to discuss the very particular statistical status and properties which such distributions seem to entail. On this basis, we tried to justify why *semistabilty* should play a central role in the modeling of random events. This work is then organized as follows.

In Section 2, we first address the well-known problem which consists in modelling observed random events out of the celebrated *Fréchet-Weibull-Gumbel* trio, which is known to embody the only possible limiting *Fisher-Tippett* distributions for maxima of independent and identically distributed (*iid*) random variables. Some of their remarkable properties are briefly discussed, focusing in particular on the one of their *stability* under the operation of "maximum". Standard statistical methodology from parametric estimation theory is available if the data consist of a sample with distribution any of the three *max-stable* distributions (the generalised extreme value distribution in the statistical terminology): statistical inference in this case is referred to as "fitting of annual maxima" and rests upon the idea that these variables can be interpreted as maxima over disjoint time period after a crude de-clustering of the full set of data.

In Section 3, we discuss a concept whose statistical insight is indeed deeper than the one of *stability*, namely the one of *max-semistability*, leading to a larger class of *max-infinitely divisible* statistical distributions [11–13,26]. These can be defined as the fixed point of some transformation on their probability distribution function (*df*) which basically reflects the statistical *self-similarity* properties of their solutions; there seems to be much analogy of this statistical point of view with the ones of discrete scale invariance and log-periodicity arising from renormalization group theory in the Physics' literature (see [35] for a survey, and [3,17–19,32–34,37–40] for applications of these ideas to real-world problems). More precisely, *max-semistable* observable, as random events, are identified with the ones whose *df*, say F, satisfies a functional equation of the form

$$F\left(v\right)=F\left(\frac{v}{c}+\beta\right)^{\gamma}$$

for some $c>0$, $\beta\in\mathbb{R}$, $\gamma\geq 1$. Forcing $\beta=0$ in this functional equation yields solutions whose main feature identifies with the notion of strict *max-selfsimilarity.* Allowing $\beta\neq 0$ identifies with the notion of *max-selfsimilarity* in the broad sense (or *max-semistability*), allowing for shifts to explain the observable.

There are three possible solutions to the above functional equation, each with its specific support, extending (and actually including) the three *max-stable* distributions in that their scale parameters are no longer

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constant, but rather allowed to vary in a log-periodic fashion. In our interpretation, a log-periodic scale parameter is therefore basically the signature of statistical *max-self-similarity.* These distributions stand as appealing candidates to model *observed* random events in the Natural Sciences, precisely because of their *max-selfsimilarity.* In this interpretation, the observed "global" random event interprets as the maximum of a *Poissonian* number of "local" events , in other words, as a clustering of "micro-events". What the functional equation adds to this is that this global event could as well result from *more* local events but with reduced shifted amplitudes. There is some incertitude on the way the final global result may be produced in that the exact scale, intensity and location of the observable are undetermined.

It turns out that the first two solutions of these *max-semistable* distributions may also be viewed like the exponentials of the third one in such a way that these may as well be considered as *log-max-semistable.* To complete the picture, we exhibit, in Section 4, the six possible types of *log-max-semistable* distributions, adding four statistical distributions to the two already mentioned from this class. In a similar fashion, using a symmetry trick, it is suggested that there exists three possible types of *min-semistable* distributions and six possible types of *log-min-semistable* distributions, each with specific statistical properties.

It is a very common feature in Physics that *observable* are modeled through the logarithms of other quantities with much more "appealing" meaning, such as the "energy" or intensity of some underlying phenomenon. The distinctive feature of the logarithmic scale for observable is that it measures the distance between two intensities through their ratio rather than their difference; this transformation thus supplies a discrimination power between two physical signals which is insensitive to their absolute intensities in that it essentially deals with their ratio. In our context, this means that *max-* or *min-semistable* variables are to be considered as the observable of some "hidden" physical phenomenon, which therefore proves itself *log min-*(or *max*)*-semistable.* Consequently, it is argued that *log min-*(or *max*)*-semistable* distributions should stand for most relevant statistical models for the intensity of random events measured in logarithmic scale.

In Section 5, we finally focus on a particular *log min-semistable* model which exhibits some sort of a "*Pareto*" critical behavior when the structure parameter crosses the value one. In this model, the tails of the energy variable are at least moderately heavy (ranging from moderately to extremely heavy, through heavy in a *Pareto*-like sense) whereas the ones of its logarithm (the associated observable) are at most moderately heavy, emphasizing the fact that observable generically exhibit tails much thinner than the ones of the underlying hidden variable. This fact and the logarithmic scale arguments suggest that when dealing with a sequence of intensities, if one is to understand an "average" event, one should rather work either with the maximum term of the sequence or use geometrical averaging since the standard arithmetic average may be extremely ill-defined.

2 Max-stable models for random events

We first recall some salient facts concerning the *Fréchet-Weibull-Gumbel* models, *i.e.* with limit laws of classical extreme value theory. These models are known to constitute the limit distributions for centred and normalised *iid* maxima: in this context, they are to maxima what *Lévy*-stable laws are to sums. Their maximum domains of attraction and the centring and normalising constants are available for example in [6,31]. An introductory treatment of extreme value theory may be found, among others, in *Lamperti*'s paperback on probability theory [21].

2.1 The Fréchet model of max-stability

Consider the class of positive random variables, defined through

$$V^+ = (S/s)^{-1/a}, \ a > 0 \text{ and } s > 0 \tag{1}$$

where S is an exponentially distributed random variable with mean unity, *i.e.* with *df*

$$F_S(s) = 1 - \exp(-s). \tag{2}$$

The variable V^+ can be seen as the output of some deterministic "machine", with parameters (s, a), triggered by some stochastic source of disorder S. Note that in the language of statistical physics, the source S is the random variable with maximum entropy under the constraint that its average value is equal to one.

While s is simply a scaling factor, the "structure" parameter a defines, roughly speaking, the way in which the disorder generated by the source S is concentrated through the transformation (1) over the positive real axis. For positive v, the probability density function (*pdf*) and *df* of V^+ are obtained easily by combining (1) with (2), yielding the *Fréchet distribution:*

$$f_{V^+}(v) = asv^{-(1+a)} \exp\left(-sv^{-a}\right), \ v > 0$$

$$F_{V^+}(v) = \exp\left(-sv^{-a}\right), \ v > 0. \tag{3}$$

Let us now recall [7,31] that a distribution is said to be *regularly varying* it there exists some finite strictly positive constant a (the tail exponent) such that its complementary probability distribution function (*cdf*) satisfies

$$\overline{F}(v) \underset{v\uparrow+\infty}{\sim} v^{-a} L(v) \tag{4}$$

where L is some slowly varying function, *i.e.* such that for all strictly positive t:

$$\lim_{v\uparrow+\infty} \frac{L(tv)}{L(v)} = 1. \tag{5}$$

Such distributions have only moments of order strictly less than a.

Clearly, for the *Fréchet* variable V^+

$$\overline{F}_{V^+}(v) = 1 - \exp\left(-sv^{-a}\right) \underset{v\uparrow+\infty}{\sim} sv^{-a} \tag{6}$$

so that the *Fréchet* distribution F_{V^+} is regularly varying, with tail exponent a. The variable V^+ only possesses moments of order less than a. Note that when $a \in (0,1)$, V^+ does not even have a mean value, *i.e.*, with E the symbol for mathematical expectation, $E(V^+) = +\infty$.

Concerning V^+, it may be shown [15] that

$$E\left(\left[V^+\right]^\beta\right) = s^{\beta/a}\Gamma(1-\beta/a) \tag{7}$$

so that β-moments for V^+ are finite as soon as $\beta < a$. Here Γ is *Euler*'s function. If $a > 1$, the mean value m_{V^+} of V^+ is thus

$$m_{V^+} = s^{1/a}\Gamma(1-1/a).$$

Concerning its median value, defined by $F_{V^+}(\overline{m}_{V^+}) = 1/2$, it is: $\overline{m}_{V^+} = \left(\frac{1}{s}\log 2\right)^{-1/a}$. Note also that the mode of V^+ is always non-zero and is $m^*_{V^+} = \left(\frac{a+1}{as}\right)^{-1/a}$. Actually, the class of all *Fréchet* distributions can be obtained while allowing a shift of V^+. Let $x^+ \in \mathbb{R}$. Consider the shifted variable $X^+ := V^+ + x^+$. From (3), the shifted variable now admits the *df*

$$F_{X^+}(x) = \exp -s\left(x-x^+\right)^{-a}, \; x > x^+ \tag{8}$$

which is the *Fréchet "max-stable"* model (see Sect. 2.4 for a justification of this terminology) with support $(x^+, +\infty)$, unbounded to the right.

2.2 The Weibull model of max-stability

Next consider the negative *inverse* random variable $V^- := -1/V^+$. It has *pdf* and *df* given by

$$\begin{aligned} f_{V^-}(v) &= as(-v)^{a-1}\exp\left(-s(-v)^a\right), \; v<0 \\ F_{V^-}(v) &= \exp\left(-s(-v)^a\right), \; v<0 \end{aligned} \tag{9}$$

and is identified with the *Weibull* distribution.

The *Weibull* random variable V^- is special case of the so-called Von Mises variables [6], whose *df* can be written in the form

$$F_{V^-}(v) = F_{V^-}(v_0)\exp\left[-\int_v^{v_0} h_{V^-}(v)\,\mathrm{d}v\right], \; v < v_0 < 0 \tag{10}$$

and where the (positive) *hazard energy density* h_{V^-} defined by this formula verifies

$$\lim_{v\downarrow-\infty} -vh_{V^-}(v) = +\infty. \tag{11}$$

The *df* of a *Von Mises* variable decreases towards zero faster than any power-law, so that these distributions are light-tailed at $-\infty$ (or rapidly varying). As a consequence, the variable $-V^-$ has moments of any arbitrary positive order. If in addition the function h_{V^-} verifies

$$\lim_{v\downarrow-\infty} h_{V^-}(v) = 0 \tag{12}$$

the variable V^- is said to have moderately heavy left tail. Otherwise, it is super-exponential, as it has tail lighter than exponential ones.

In the *Weibull* example, we get, as v gets close to $-\infty$

$$h_{V^-}(v) \sim as(-v)^{a-1}. \tag{13}$$

Thus, when $0 < a < 1$ the *Weibull* variable V^- has moderately heavy left tail, whereas for $a \geq 1$ it has super-exponential thin left tail. From (7), we clearly get

$$E\left(\left[-V^-\right]^\beta\right) = s^{-\beta/a}\Gamma(1+\beta/a). \tag{14}$$

Hence, β-moments for $-V^-$ exist as soon as $\beta > -a$; in particular the mean value of V^- is

$$m_{V^-} = -m_{-V^-} = -s^{-1/a}\Gamma(1+1/a). \tag{15}$$

Note that whereas the mean value of V^- is given by (15), its median value, say $\overline{m}_{V^-}$, defined as the solution of $F_{V^-}(\overline{m}_{V^-}) = 1/2$, is

$$\overline{m}_{V^-} = -\overline{m}_{-V^-} = -\left(\frac{1}{s}\log 2\right)^{1/a} = -1/\overline{m}_{V^+}. \tag{16}$$

Finally, the distribution of V^- has a non-zero mode, say $m^*_{V^-}$, at the only condition that $a > 1$, and if this is the case

$$m^*_{V^-} = -m^*_{-V^-} = -\left(\frac{a-1}{as}\right)^{1/a}. \tag{17}$$

Actually, the class of all *Weibull* distributions can be obtained while allowing a shift of V^-. Let then $x^- \in \mathbb{R}$. Consider the shifted variable $X^- := V^- + x^-$. From (9), the shifted variable now admits the *df*

$$F_{X^-}(x) = \exp -s\left(x^- - x\right)^a, \; x < x^- \tag{18}$$

which is the *Weibull* max-*stable* model (see Sect. 2.4 for a justification of this terminology) with support $(-\infty, x^-)$, unbounded to the left.

2.3 The Gumbel model: max-stability for real-valued random variables

Let us introduce the variable

$$X = \log V^+ \tag{19}$$

i.e. the logarithm of the *Fréchet* variable V^+.

Let us then compute the distribution of X and underline some of its remarkable properties.

From (6, 19) the *df* for the variable X is found to be

$$F_X(x) = \exp\left(-s\mathrm{e}^{-ax}\right). \tag{20}$$

We identify this *df* with the one of a *Gumbel* distribution [9,14]: the "exp-*Fréchet*" variable X is a *Gumbel* variable on the extended domain $(-\infty, +\infty)$.

For any choice of (s, a), the variable X is *Von Mises*'; in addition, it is super-exponential, which means that the tails of its *df* decrease towards zero at exponential rate or faster at both extremities $\pm\infty$ of the support. Hence, the distribution is "thin", although very asymmetric (exponential at $x = +\infty$ and doubly exponential at $x = -\infty$): taking the logarithm of V^+ thins its tails in a drastic way. From (7), the *Laplace* transform $Z_X(\beta)$ of X is given by

$$Z_X(\beta) := E\left(\mathrm{e}^{\beta X}\right) = E\left(\left[V^+\right]^\beta\right) = s^{\beta/a}\Gamma(1-\beta/a). \tag{21}$$

This function is thus defined on the range $\beta < a$, therefore containing the origin $\beta = 0$, as required. As a result, in sharp contrast to the *Fréchet* distributed variable V^+ itself, the variable X always has convergent moments of arbitrary integral order, which can be obtained as the *Taylor* coefficients of $Z_X(\beta)$ at $\beta = 0$. For instance, denoting as γ the *Euler*'s constant, the mean value of X is found to be

$$m_X = E(X) = \frac{1}{a}(\log s + \gamma) \simeq \frac{1}{a}(\log s + 0.5772). \tag{22}$$

In addition, the median value of X is

$$\overline{m}_X = \frac{1}{a}(\log s - \log\log 2) \simeq \frac{1}{a}(\log s + 0.3665). \tag{23}$$

It should be emphasized that the mean and median have a simple expression in terms of the pair (s, a). In addition, (see *e.g.* [8] for an exploitation of this fact) the distribution X is always unimodal, and even strongly unimodal, which means that the information (log-density) function $I_X(x) := -\log f_X(x)$ is strictly convex; its mode is

$$m_X^* = \frac{1}{a}\log s. \tag{24}$$

A remarkable feature of the mean-median-mode trio in the *Gumbel* model is thus that

$$m_X > \overline{m}_X > m_X^*. \tag{25}$$

2.4 Max-stable models as limit laws in Statistics

Fréchet, *Weibull* and *Gumbel* distributions are all *max-stable* in the following sense. Let X be any of these variables with *df* either given by (8, 18, 20). Then, for any $n \geq 1$, there exists two sequences $x_n \in \mathbb{R}$, $\sigma_n > 0$ such that, with $X_m \overset{d}{=} X$, $m = 1, ..., n$, *iid* random variables

$$X \overset{d}{=} \max_{m=1,\ldots,n} \frac{X_m - x_n}{\sigma_n}.$$

Here the symbol $\overset{d}{=}$ means that the random variables share the same distributions. This class is a proper subclass of the one of max-infinitely divisible (MID) variables X for which, for any $n \geq 1$, with $\mathcal{X}_{m,n} \overset{d}{=} \mathcal{X}_n$, $m = 1...; n$

$$X \overset{d}{=} \max_{m=1,\ldots,n} \mathcal{X}_{m,n}.$$

A characteristic criterion for MID random variables X is thus that for any $n \geq 1$, $F_X(x)^{1/n}$ must be a *df* (the one of $\mathcal{X}_n$); alternatively, for any $t > 0$, $F_X(x)^t$ must be a *df*. It turns out that *all* unidimensional distributions may be shown to be MID so that this notion is fully meaningful in higher dimensions only [5].

Turning back to *max-stable* variables, a consequence of their properties is that they appear as limit laws in the *statistics of extremes* problem in the following sense: if X is *max-stable*, then there exists a random variable $\mathcal{X}$ and two sequences $x_n \in \mathbb{R}$, $\sigma_n > 0$ such that

$$\max_{m=1,\ldots,n} \frac{\mathcal{X}_m - x_n}{\sigma_n} \overset{d}{\to} X \text{ as } n \uparrow +\infty$$

where $\mathcal{X}_m \overset{d}{=} \mathcal{X}$, $m \geq 1$ is an *iid* sequence. The variable $\mathcal{X}$ is said to belong to the max domain of partial attraction (MDPA) of X (note that X itself belongs to its own MDPA). Thus, *max-stable* distributions derive their importance from the fact that they are the limit laws of the maximum of *iid* random variables $\mathcal{X}_m$, $m \geq 1$, after a convenient location-scale transform [6,9,10,14]. Thus, for maxima, the max-stable laws play the role which *Lévy-stable* probability distributions play for sums in the central limit theorem.

3 The max-semistable models and max-selfsimilarity

We now discuss a concept whose generality is larger than the one of *stability*, namely the one of *max-semistability* [11–13,26,27]. *Max-semistable* laws are identified with the ones whose *df* satisfies a functional equation of the form

$$F(v) = F\left(\frac{v}{c} + \beta\right)^\gamma \tag{26}$$

for some $c > 0$, $\beta \in \mathbb{R}$, $\gamma \geq 1$. They constitute the "*max* version" of the notion of *semistability* for sums first introduced by *Lévy* in 1937 (see [23] p. 45 for a survey on this point) and worked out in [22].

3.1 Max-semistability of type one

Let $\gamma > 1$ and $c \in (0, 1)$. First, consider the class of positive random variable V^+ whose *df* satisfies the simpler functional equation

$$F_{V^+}(v) = F_{V^+}(v/c)^\gamma. \tag{27}$$

These variables will be identified with the so-called *max-semistable* variables and may thus be seen as the fixed point of some transformation to be elucidated below as statistical *self-similarity*, in the strict sense.

Upon reasoning with the function $H(x) := -\log F_{V^+}(e^x)$, this functional equation takes the simpler form $H(x) = \gamma H(x - \log c)$; letting $H(x) := e^{-ax} P(x)$, the class of solutions of (27) are then easily found to be, formally

$$F_{V^+}(v) = \exp -v^{-a} P(\log v) \tag{28}$$

where $a > 0$ is uniquely defined through

$$\gamma c^a = 1 \tag{29}$$

and where $P(x)$ is a periodic (with period $-\log c$) function on the real line, that is satisfying $P(x) = P(x - \log c)$.

As F_{V^+} must be the *df* of some random variable V^+, additional conditions have to be imposed on P. First P has to be non-negative: we shall then let $P(x) := e^{a\nu_c(x)}$, for some periodic function ν_c. Next, $x - \nu_c(x)$ has to be a non-decreasing function, in such a way that the hazard function $v^{-a} e^{a\nu_c(\log v)}$ be non-increasing with v. Finally, it is necessary that $\nu_c(x)$ be right-continuous and bounded. Conversely, if function P satisfies all these conditions, then the function F_{V^+} defined by (28) is a *df* and satisfies the functional equation (27).

Note also that the functional equation (27) is also

$$F_{V^+}(v) = F_{V^+}(v/\overline{c})^{\overline{\gamma}} \tag{30}$$

with $\overline{c} := 1/c > 1$ and $\overline{\gamma} := 1/\gamma \in (0,1)$ so that $\nu_c = \nu_{\overline{c}}$ is an additional requirement that ν_c should meet, observing that $\overline{a} = \log \overline{\gamma} / -\log \overline{c} = a$.

A most simple and fundamental example of such ν_c is, with $\rho \in (0,1]$

$$\nu_c(x) = \frac{\rho \log c}{2\pi} \sin \frac{2\pi x}{-\log c} \tag{31}$$

assuming a single term in the full *Fourier* series expansion of $\nu_c(x)$.

To summarize, a *max-semistable* model for $V^+ > 0$ is

$$F_{V^+}(v) = \exp -s_{V^+}(v)\, v^{-a} \tag{32}$$

where the log-scale parameter is allowed to vary periodically with $\log v$ according to

$$\log s_{V^+}(v) = a\nu_c(\log v). \tag{33}$$

In a *max-semistable* model the scale parameter is thus log-periodic. Note that in the example (31), $s_{V^+}(v) \in \left[\gamma^{-\rho/(2\pi)}, \gamma^{\rho/(2\pi)}\right]$, so that the value 1 lies in the interval.

Let us stress some additional properties of *max-semistable* positive variables

- We start with a notion of statistical *max-selfsimilarity.* Let $X := X(1)$ be a positive random variable with *df* F_X. With $\varsigma > 0$, define the variable $X(\varsigma)$ as the variable whose *df* is $F_{X(\varsigma)} = F_X^{\varsigma}$, raising F_X to the power ς (in our univariate context, F_X^{ς} is always the *df* of some random variable; this property is the one of *max-infinite divisibility* of X). The variable X will be said statistically *max-selfsimilar*, with exponent H, if the following holds for some $\varsigma > 0$

$$X(\varsigma) \stackrel{d}{=} \varsigma^H X(1). \tag{34}$$

Now, the functional equation (27) means that

$$V^+(\gamma) \stackrel{d}{=} c^{-1} V^+. \tag{35}$$

With $a > 0$ defined through (29), this is also

$$V^+(\gamma) \stackrel{d}{=} \gamma^{1/a} V^+. \tag{36}$$

Thus the class of all such V^+ identifies with the class of statistically *max-selfsimilar* positive random variables, with exponent $H = 1/a > 0$. By its construction, a positive *max-semistable* variable is in the *max-infinitely divisible* class.

- The physical reason why one should be interested in such *self-similar* variables proceeds as follows: as V^+ is a *max-infinitely divisible* variable, this basically means that, "roughly" speaking (this reasoning can be made rigorous)

$$V^+ = \max_{m=1\ldots P(1)} \varepsilon_m \tag{37}$$

where $P(1)$ is a *Poisson* variable with mean value 1. Here $(\varepsilon_m)_{m\geq 1}$ is an *iid* sequence of "local" events. The observed "global" random event V^+ is thus assumed to be the maximum of a *Poissonian* number of "local" events which sounds reasonable, physically: the observed variable is a clustering of "micro-events".
What the functional equation (27) tells us, in addition, is that this global event V^+ could as well result from *more* local events (replacing $P(1)$ by $P(\gamma)$, $\gamma > 1$, in (37)) but with *smaller* reduced amplitudes, (substituting $c\varepsilon_m$ to ε_m, $m \geq 1$, in (37)), in such a way that V^+ is also, for some judicious constants c and γ

$$V^+ = \max_{m=1\ldots P(\gamma)} c\,\varepsilon_m.$$

It is some sort of an ignorance principle on the way the final global result may be produced: the exact scale and intensity of the observable are undetermined. As a result, there are now *two* basic unknown structure parameters, namely c and γ (or alternatively c and $a = \log\gamma / -\log c$) to deal with.
This illustrates the commonly accepted fact that physical variables of interest are "invariants" in some statistical sense which is made precise here.
We finally underline the analogy of the problem treated in this monograph on a statistical basis and the one of *Sornette* and coworkers on discrete scale

invariance arising from Renormalization Group theory in Physics (see [35] and the references therein for a survey). There, the exponent a is naturally interpreted as a "dimension" and the log-periodic decorations lead to the richer notion of a complex dimension, whereas, in the statistical language advocated here, a simply is a tail exponent and the log-periodic decorations states that the underlying stable variable possibly exhibits an oscillating scale parameter. In any case, there is empirical evidence of log-periodicity in diverse application fields among which: finance [17–19,40], turbulence [39], rupture theory [3], DLA growth [38], geophysics [33,34,37] and frustated systems' statistics [32].

- Note that a constant function $\nu_c(x) = \nu$ of (28) satisfies all the requirements, so that a *max-stable* model is a particular case of a *max-semistable* model. *Max-semistability* identifies here with the notion of *max-selfsimilarity*, which is weaker than *max-stability*, leading to a larger class of distributions.
- From $\overline{F}_{V^+}(v) \sim v^{-a} \mathrm{e}^{a\nu_c(\log v)}$. Thus V^+ is "close" to be regularly-varying with tail index $a > 0$, just like the *Fréchet* model was. In fact, although $L(v) := \mathrm{e}^{a\nu_c(\log v)}$ is not slowly varying, it satisfies the weaker condition that, for all $t > 0$, $L(tv)/L(v)$ has a liminf and a limsup for large v.
- Empirical evidence of *max-semistability* goes through the observation that

$$-\log\left(-\log F_{n,V^+}(v)\right) = a\left(\log v - \nu_c(\log v)\right) \quad (38)$$

i.e. that a plot of $-\log\left(-\log F_{n,V^+}(v)\right)$ against $\log v$ should exhibit oscillations around a linear trend with positive slope a. Here, F_{n,V^+} is the empirical *df* that can be obtained from an *n-iid* sample of V^+. Note also that the theoretical *df* of V^+ is such that, with $k \in \mathbf{Z}$, $F_{V^+}\left(c^{-k}\right) = \exp - sc^{ak}$ with $\log s = a\nu_c(0)$: evaluating F_{V^+} at geometrically scattered points c^{-k}, $k \in \mathbf{Z}$, the oscillating part vanishes.

Actually, the variable V^+ does not cover all the class of *max-semistable* variables. Those obtained after a shift of V^+ are also in this class as we now show.

Let $x^+ \in \mathbb{R}$. Consider the shifted variable $X^+ := V^+ + x^+$. The shifted variable now satisfies the functional equation of the type (26), with $\beta = x^+(1 - 1/c)$

$$F_{X^+}(x) = F_{X^+}\left(\left(x - x^+\right)/c + x^+\right)^\gamma, \; x > x^+ \quad (39)$$

whose solution is

$$F_{X^+}(x) = \exp - \left(x - x^+\right)^{-a} \mathrm{e}^{a\nu_c\left(\log\left(x - x^+\right)\right)}, \; x > x^+. \quad (40)$$

This accounts for the *max-semistable* model of type one. In this formulation, the observed "global" random event X^+ still is the maximum of a *Poissonian* number of "micro-events", as in (37). However, from the functional equation (39), this global event X^+ could as well result from more local events but with reduced *and* shifted amplitudes, in constants c, γ and β

$$X^+ = \max_{m=1\ldots P(\gamma)} c\left(\varepsilon_m - \beta\right).$$

The exact way the final global result may be produced is unknown in that the exact scale, intensity *and* location of the observable are now basically undetermined. Thus *max-semistability* extends the notion of strict *max-selfsimilarity* discussed above in that the location parameter of the observable is also unknown: in this sense, it is a "broad sense" *max-selfsimilarity*.

3.2 Max-semistability of type two

We now briefly discuss a second class of *max-selfsimilar* random variables.

Letting $V^- := -1/V^+ < 0$, its *df* is, from (28)

$$F_{V^-}(v) = F_{V^+}(-1/v) = \exp - s_{V^-}(v)(-v)^a \quad (41)$$

with scale function

$$s_{V^-}(v) := s_{V^+}(-1/v). \quad (42)$$

This variable is in the *Von Mises* class, just like *Weibull* was. It is again *max-semistable* and negative in that it now satisfies the functional equation

$$F_{V^-}(v) = F_{V^-}(vc)^\gamma. \quad (43)$$

Note that this functional equation means that

$$V^-(\gamma) \stackrel{d}{=} cV^-. \quad (44)$$

With $a > 0$ defined through (29), this is also

$$V^-(\gamma) \stackrel{d}{=} \gamma^{-1/a} V^-. \quad (45)$$

Thus the class of all such $V^- < 0$ identifies with the class of statistically *max-selfsimilar* random variables, with exponent $H = -1/a < 0$.

Again, this set of variables does not cover all the possible range. Rather, let $x^- \in \mathbb{R}$ and consider the shifted variable $X^- := V^- + x^-$. The shifted variable now satisfies the functional equation of the type (26)

$$F_{X^-}(x) = F_{X^-}\left(\left(x - x^-\right)c + x^-\right)^\gamma, \; x < x^- \quad (46)$$

whose solution is

$$F_{X^-}(x) = \exp - \left(x^- - x\right)^a s_{V^-}\left(x - x^-\right), \; x < x^-. \quad (47)$$

This model stands for the *max-semistable* distribution of the second type.

3.3 The max-semistable model of type three: "extended" Gumbel

Finally, we proceed with the construction of a last class of *max-semistable* variables. Let us introduce the variable

$$X = \log V^+ \tag{48}$$

i.e. the logarithm of the *max-semistable* (and *max-selfsimilar*) variable V^+ defined in (28). Let us then compute the distribution of X and underline some of its remarkable properties.

From (28, 48) the *df* for the variable X is found to be

$$F_X(x) = \exp\left(-s_X(x)\,\mathrm{e}^{-ax}\right) \tag{49}$$

with

$$\log s_X(x) = \log s_{V^+}(\mathrm{e}^x) = a\nu_c(x). \tag{50}$$

We identify this *df* as the one of an extended *Gumbel* distribution [14] as the scale parameter is now allowed to vary log-periodically: the "exp-*max-selfsimilar*" variable X is an extended *Gumbel* variable on the domain $(-\infty, +\infty)$.

This variable X is *Von Mises*'; in addition, it is super-exponential, which means that the tails of its *df* decrease towards zero at exponential rate or faster at both extremities $\pm\infty$ of the support. Hence, the distribution is "thin", although very asymmetric (exponential at $x = +\infty$ and doubly exponential at $x = -\infty$).

With $\gamma > 1$ and $\beta := -\log c > 0$, we note that X is a real-valued random variable, whose *df* is the solution to the functional equation of the type (26)

$$F_X(x) = F_X(x+\beta)^\gamma,\ x \in \mathbb{R} \tag{51}$$

where

$$a = \log \gamma/\beta. \tag{52}$$

Thus X is itself a *max-semistable* variable with support $\mathbb{R}$. From this observation, the *max-semistable* variables $V^+ := \exp X$ and $V^- := -\exp -X$ can also be interpreted as the exponentials of the *max-semistable* variable X and therefore also belong to the *log-max-semistable* models' class.

Note also that the random variable $X = \log X^+$ (substituting X^+ to V^+ in (48)) satisfies a functional equation of the above type $F_X(x) = F_X(x+\beta)^\gamma$, $x \in \mathbb{R}$, but now with $\beta = -\mathrm{e}^{x^+}\log c > 0$; a similar equation may be found for $-\log(-X^-)$. The remarkable point about (X^+, X^-) is therefore that they may be interpreted as *max-semistable* variables, but also like the exponentials of a *max-semistable Gumbel* variable: they are thus simultaneously *max-semistable* and *log-max-semistable*. This suggests that *log-max-semistable* variables are also of special interest in our context; in the sequel, we shall then exhibit all six possible types of *log-max-semistable* variables.

Before proceeding, let us supply an additional remark, underlining the importance of *max-semistable* models in Statistics [25].

3.4 Max-semistable models as limit laws in Statistics

Let X be any *max-semistable* variables with *df* either given by (40, 47, 49). These variables also appear as limit laws in the *statistics of extremes* problem in the following wider (geometrical) sense: if X is *max-semistable*, then, as it can easily be checked, (see also [11,13]), there exists a random variable $\mathcal{X}$ and three sequences $x_n \in \mathbb{R}$, $\sigma_n > 0$ and $\gamma_n > 0$ such that

$$\max_{m=1,\ldots,\gamma_n} \frac{\mathcal{X}_m - x_n}{\sigma_n} \overset{d}{\to} X \text{ as } n \uparrow +\infty \tag{53}$$

where $\mathcal{X}_m \overset{d}{=} \mathcal{X}$, $m \geq 1$ is an *iid* sequence. The integer-valued sequence γ_n is assumed to satisfy the additional geometrical growth properties: $\lim_{n\uparrow+\infty} \gamma_n = +\infty$ and $\lim_{n\uparrow+\infty} \gamma_{n+1}/\gamma_n = \gamma \geq 1$.

The variable $\mathcal{X}$ is said to belong to the max domain of partial attraction (MDPA) of X (note that again X itself belongs to its own MDPA).

Thus, *max-semistable* distributions derive their importance in Statistics from the fact that they are the limit laws of the maximum of a *geometric* number γ_n of *iid* random variables $\mathcal{X}_m$, $m \geq 1$, after a convenient location-scale transform. *Max-semistable* variables also form a proper subclass of MID variables. *Max-stable* variables are *max-semistable* and can be obtained in the limit $\gamma \to 1^+$. This latter class is therefore an extension of the former one, hence its name.

Remark 1 Assuming for simplicity γ to be an algebraic number, integral sequences with the required geometrical growth properties may be produced (for example) in the following "canonical" way. Let $K > 1$ be some integer. Let $(a_1, .., a_K)$ be a $K-$vector of integers such that $a_K \neq 0$ and satisfying $\sum_{k=1}^{K-1} a_k > 0$. Define the $K \times K$ irreducible (and even primitive) matrix A as

$$A := \begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{K-1} & 0 & 0 & & 1 \\ a_K & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

With "prime" denoting transposition of vectors, let the $K-$column vector $\mathbf{N}_0$ be defined as $\mathbf{N}_0' := (1, 0, ..., 0)$. Define now the integer-valued vectors sequence $(\mathbf{N}_n,\ n \geq 0)$ through the recurrence

$$\mathbf{N}_{n+1} = A\mathbf{N}_n,\ n \geq 0.$$

Then, with $\mathbf{N}_n(k)$ denoting the kth entry of $\mathbf{N}_n$, $\gamma_n := \sum_{k=1}^{K} \mathbf{N}_n(k)$ exhibits the geometrical growth property where $\gamma > 1$ is the spectral radius of A (which is here the simple largest eigenvalue of A). This is a "branching" interpretation of the way the sequence $(\gamma_n,\ n \geq 0)$ could be produced: indeed, suppose an individual of "type" $k = 1$

initially generates a_k individuals of type k, $k = 1, ..., K$. At subsequent steps $n > 1$, iterate this process observing that for $k = 2, ..., K$, individuals of type k generates a single individual of type $k-1$. The number $\mathbf{N}_n(k)$ represents the number of individuals of type k at resolution n in this branching process, and γ_n the total amount of offspring, whatever their type. If a *max-semistable* variable is to be produced in the limit as in (53), data must be aggregated in geometrical packets which may be understood from a simple tree-like structure.

4 The log-max-semistable models

It has just been emphasized that (X^+, X^-) were both *max-semistable* and *log-max-semistable*. In order to exhibit all possible types of *log-max-semistable* variables, it is convenient to study the exponentials of (X^+, X^-) now viewed as *max-semistable* variables. This operation adds four additional types of *log-max-semistable* variables, to be added to (X^+, X^-) which are indeed *log-max-semistable* variables themselves.

4.1 Log-max-semistability of type one

We shall thus be interested in the models

$$X_1^+ = \exp X^+ > \mathrm{e}^{x^+} > 0 \tag{54}$$

and by

$$X_2^+ = -\exp -X^+ \in \left(-\mathrm{e}^{-x^+}, 0\right). \tag{55}$$

As a result, (54, 55) now define two variates which are *log-max-semistable* distributed in the sense that $\log X_1^+$ and $-\log(-X_2^+)$ simply are *max-semistable* distributed. These models will be shown to present a certain number of interesting new features.

The *df* of X_1^+ and X_2^+ are now obtained easily by combining (54, 55) with (40), yielding the two *log-max-semistable* distributions of type one:

$$F_{X_1^+}(x) = F_{X^+}(\log x), \; x > x_1^+ := \mathrm{e}^{x^+} \tag{56}$$

and

$$F_{X_2^+}(x) = F_{X^+}(-\log(-x)), \; 0 > x > x_2^+ := -\mathrm{e}^{-x^+}. \tag{57}$$

Let us here stress the following point. Combining (54) with (40), we get the tail equivalence

$$\overline{F}_{X_1^+}(x) \underset{x\uparrow+\infty}{\sim} \log\left(x/x_1^+\right)^a \mathrm{e}^{a\nu_c\left(\log\log\left(x/x_1^+\right)\right)}. \tag{58}$$

From this expression, it turns out that the random variable X_1^+ has tails heavier than the ones of any power law in the sense that, for any power-law tail index $b > 0$

$$\frac{\overline{F}_{X_1^+}(x)}{x^{-b}} \underset{x\uparrow+\infty}{\rightarrow} +\infty.$$

Such distributions are said to be with tail index zero or with very heavy tails. As a result, they exhibit no finite moment of any arbitrary positive order: the exponential of X^+ has tail much fatter than the ones of X^+ itself which are with tail index $a > 0$.

4.2 Log-max-semistability of type two

We shall next introduce the models

$$X_1^- = \exp X^- \in \left(0, \mathrm{e}^{x^-}\right) \tag{59}$$

and

$$X_2^- = -\exp -X^- \in \left(-\infty, -\mathrm{e}^{-x^-}\right). \tag{60}$$

The *df* of X_1^- and X_2^- are now obtained easily by combining (59, 60) with (47), yielding the two *log-max-semistable* distributions of type two:

$$F_{X_1^-}(x) = F_{X^-}(\log x), 0 < x < x_1^- := \mathrm{e}^{x^-} \tag{61}$$

and

$$F_{X_2^-}(x) = F_{X^-}(-\log(-x)), \; x < x_2^- := -\mathrm{e}^{-x^-} \tag{62}$$

There are thus *six* possible types of *log-max-semistable* distributions, namely, the ones of $(X^+, X^-, X_1^+, X_2^+, X_1^-, X_2^-)$. Among these, the first two (X^+, X^-) play a special role as they are *also* simply *max-semistable.*

Remark 2 The *df* of the variables $(X_1^+, X_2^+, X_1^-, X_2^-)$ are themselves the solutions of a functional equation which may be derived from the ones (39, 46) that the *dfs* of (X^+, X^-) themselves verify. For example, $F_{X_1^+}(x)$ is the solution to $F_{X_1^+}(x) = F_{X_1^+}\left(x_1^+ (x/x_1^+)^{1/c}\right)^\gamma$.

4.3 Log-max-semistable models as limit laws under power normalization in Statistics

Let Z be any *log-max-semistable* variables $(X^+, X^-, X_1^+, X_2^+, X_1^-, X_2^-)$. For each such variable, $Z^+ = \exp X$, or $Z^- = -\exp -X$, with X *max-semistable* for which, following (53), there exists a random variable $\mathcal{X}$ in their MDPA and three sequences $x_n \in \mathbb{R}$, $\sigma_n > 0$ and a geometrical series $\gamma_n > 0$ such that

$$\max_{m=1,...,\gamma_n} \frac{\mathcal{X}_m - x_n}{\sigma_n} \xrightarrow{d} X \text{ as } n \uparrow +\infty \tag{63}$$

where $\mathcal{X}_m \overset{d}{=} \mathcal{X}$, $m \geq 1$ is an *iid* sequence. As a result, for any Z log-max-semistable of the form Z^+, there exists a random variable $\mathcal{Z} := \mathrm{e}^{\mathcal{X}} > 0$ in their MDPA and three sequences $z_n > 0$, $\sigma_n > 0$ and $\gamma_n > 0$ such that, with $z_n = \exp x_n$,

$$\max_{m=1,\ldots,\gamma_n} \left(\frac{\mathcal{Z}_m}{z_n}\right)^{1/\sigma_n} \overset{d}{\to} Z^+ \text{ as } n \uparrow +\infty. \quad (64)$$

In a similar way, for any Z log-max-semistable of the form Z^-, there exists a random variable $\mathcal{Z} := -\mathrm{e}^{-\mathcal{X}} < 0$ in their MDPA and three sequences $z_n < 0$, $\sigma_n > 0$ and $\gamma_n > 0$ such that, with $z_n = -\exp -x_n$,

$$65 \max_{m=1,\ldots,\gamma_n} -\left(\frac{\mathcal{Z}_m}{z_n}\right)^{1/\sigma_n} \overset{d}{\to} Z^- \text{ as } n \uparrow +\infty. \quad (65)$$

Thus, *log-max-semistable* distributions derive their importance from the fact that they are the limit laws of the maximum of a *geometric* number γ_n of *iid* random variables $\mathcal{Z}_m$, $m \geq 1$, after a convenient *power normalization* [13,30].

Remark 3 Similarly, using a symmetry "trick", the six variables

$$\begin{aligned}&\big(X_+ := -X^-, X_- := -X^+, X_{+,1} := \mathrm{e}^{X_+},\\ &\quad X_{+,2} := -\mathrm{e}^{-X_+}, X_{-,1} := \mathrm{e}^{X_-}, X_{-,2} := -\mathrm{e}^{-X_-}\big) \quad (66)\end{aligned}$$

are all *log-min-semistable*. With $x_+ = -x^-$, $x_- = -x^+$, $x_{+,1} = \mathrm{e}^{x_+}$, $x_{+,2} = -\mathrm{e}^{-x_+}$, $x_{-,1} = \mathrm{e}^{x_-}$, $x_{-,2} = -\mathrm{e}^{-x_-}$, their respective support is

$$(x_+, \infty),\ (-\infty, x_-), (x_{+,1}, \infty), \\ (x_{+,2}, 0), (0, x_{-,1}),\ (-\infty, x_{-,2})$$

and their *df* may easily be deduced from (40, 47, 66). The *(log)-max-semistable*variables were characterized by functional equations for their *dfs*; concerning *(log)-min-semistable*, they are rather characterized by functional equations for their *cdfs*.

To take an example, one may easily check that

$$\begin{aligned}\overline{F}_{X_+}(x) &= F_{X^-}(-x)\\ &= \exp - (x - x_+)^a s_{V^-}(x_+ - x),\\ x > x_+ &= -x^- \end{aligned} \quad (67)$$

satisfies the functional equation $\overline{F}_{X_+}(x) = \overline{F}_{X_+}((x - x_+) c + x_+)^\gamma$, $x > x_+$, involving now the *cdf* $\overline{F}_{X_+}$.

5 Log-min-semistable model for energy

It is a very common feature in the natural sciences that *observable* are modeled through the logarithms of other quantities with much more "physical" meaning, such as "energy" [15,16]. The distinctive feature of the logarithmic scale is that it measures the distance between two values through their ratio rather than their difference, which amounts, when these values are close, to work with their *relative* (rather than absolute) variation; this transformation thus supplies a discrimination power between two signals which is insensitive to their absolute intensities, rather dealing with the ratio of these intensities. Thus, the intensity of noise, as perceived by the human ear, is usually measured in decibels, *i.e.* using a logarithmic scale. Similarly, earthquake magnitudes are determined from the logarithm of the amplitudes of waves recorded by seismographs; additional adjustments are included in the magnitude formula to compensate for the variation in the distance between the various seismographs and the epicenter of the earthquake, making data self-consistent.

In our context, this means that *max-* or *min-semistable* variables are to be considered as the observable of some physical phenomenon, say Z, which proves itself *log min-* (or *max*) *-semistable*. Observable are often assumed, in the applications, to take values bounded from below but unbounded from above making X^+ (as defined in (40)) and X_+ (as defined in (67)) of particular interest in this respect. Similarly, the model for the underlying energy variable which seems of special interest, for the same reason concerning their support, are X_1^+ (as defined in (54)) and $X_{+,1}$ (as defined in (66)). For the first duo (X^+, X_1^+) of variables, it was observed that the observable X^+ was always of the power-law type with exponent $a > 0$, in such a way that the hidden variable X_1^+ exhibited very heavy tails of index zero, for *any* value of this parameter. This is not the case for the second duo of variables $(X_+, X_{+,1})$ for which special attention is needed.

In the sequel, we shall therefore call $Y := X_+$ (the *min-semistable* observable) and thus focus on the particular *log-min-semistable* "energy" variable $Z := X_{+,1} = \mathrm{e}^Y$. This variable exhibits very interesting "critical" properties similar to the ones discussed in [16], in the restricted context of *log-Weibull* distributions which could account for tails ranging from moderately heavy to "very heavy" through regularly-varying tails (*i.e.* of the power-law type).

First observe that, with $y_0 := x_+ \in \mathbb{R}$, the variable Y admits the following *cdf*

$$\overline{F}_Y(y) = \exp - (y - y_0)^a s_{V^-}(y_0 - y),\ y > y_0 \quad (68)$$

which includes log-periodic corrections to the so-called "stretched exponential model" [15,20,36]. From this expression, one realizes that Y exhibits *at most* moderately heavy right tails and so presents convergent moments of arbitrary positive order. More precisely, it has super-exponential tails if and only if $a > 1$. If $a < 1$, the tails are only moderately heavy. The critical case $a = 1$ corresponds to exponential tails. In any case, Y is in the *Von Mises'* class and

$$Y = \log Z \quad (69)$$

can be interpreted as the observable of some physical hidden "energy" variable $Z > z_0 > 0$. Here $z_0 := \mathrm{e}^{y_0} > 0$ is the ground state for energy. Thus, if the observable is *semistable*, the hidden variable of physical interest is *log-semistable*.

Next, from the expression (68, 69), the *cdf* of Z is, with $s_Z(z) := s_{V^-}(-\log z/z_0) = s_{V^+}(1/\log(z/z_0))$

$$\overline{F}_Z(z) = \exp -(\log z/z_0)^a \, s_Z(z), \; z > z_0. \quad (70)$$

Note from this expression and criterion arising from (12) that the energy Z is always subexponential whatever the value of parameter a: its tails are now *at least* moderately heavy, again illustrating that the hidden variable exhibits tails much fatter than the one of the observable. Let us discuss the tails shape in more details, depending on the value of the parameter a. We shall distinguish between three cases:

– Supercritical energy model: $a > 1$.

Assuming s_Z to be derivable, when $a > 1$, it can easily be checked that, with the *hazard energy density* h_Z defined from (10) applied to (70): $\lim_{z\uparrow+\infty} zh_Z(z) = +\infty$. Thus supercritical model (70) for energy release is *Von Mises'*. In addition, $h_Z(z)$ tends to zero as $z \uparrow +\infty$, so that the tails of Z in the supercritical range are subexponential and *Von Mises'*. If $a = 2$, this model is strongly reminiscent of the one of a *log-normal* distribution whose ubiquity in Physics and elsewhere is well-known [4,2].

– Critical energy model: $a = 1$.

When $a = 1$, the *cdf* of Z is easily obtained from (70); it is:

$$\overline{F}_Z(z) = \left(\frac{z}{z_0}\right)^{-s_Z(z)}, \; z > z_0. \quad (71)$$

In some sense, this model may be viewed as a *Pareto* model but with the peculiarity that it exhibits a non constant tail index $s_Z(z)$, *i.e.* varying with z.

Remark 4 For a "true" *Pareto* model, $\overline{F}_Z(z) = \left(\frac{z}{z_0}\right)^{-s}$, $z > z_0$, with $z_0 > 0$ and $s > 0$, constant [28,41,1]. Let us recall some remarkable properties of such models. First, it may easily be checked that

$$EZ^\beta = \frac{s}{s-\beta} z_0^\beta, \text{ for } \beta < s. \quad (72)$$

Note that crossing the value $s = 1$ in such a model is critical in the sense that Z has infinite mean value when $s \leq 1$, finite when $s > 1$. Next, with $z_c > z_0$ some cut-off value, let $\widetilde{Z}_c := (Z - z_c)\,\mathbf{1}(Z > z_c) > 0$ denote the excess variable $Z - z_c$ given $Z > z_c$. Its *cdf* $\overline{F}_{\widetilde{Z}_c}(z)$, may be computed to be

$$\overline{F}_{\widetilde{Z}_c}(z) = \frac{\overline{F}_Z(z + z_c)}{\overline{F}_Z(z)} = \left(1 + \frac{z}{z_c}\right)^{-s}, \; z > 0 \quad (73)$$

independent of the characteristic scale z_0: a truncated *Pareto* distribution has no intrinsic scale but the one of the tail observer. As a result, for all $s > 0$, the conditional excess median value defined by $\overline{F}_{\widetilde{Z}_c}\left(\overline{m}_{\widetilde{Z}_c}\right) = 1/2$ is

$$\overline{m}_{\widetilde{Z}_c} = z_c\left(2^{1/s} - 1\right)$$

from which one deduces $\overline{m}_{\widetilde{Z}_c} \leq z_c$ if and only if $s \geq 1$: crossing the critical value $s = 1$ yields a median excess value smaller than the cutoff value z_c itself.

These properties are lost when allowing the scale parameter s to vary with z, as $s_Z(z)$ oscillates periodically around the critical value $s = 1$.

– Subcritical energy model: $a < 1$.

This is the most interesting feature of this model. When $a < 1$, the *cpdf* of Z satisfies the following property, as a result of (70): for any strictly positive constant b

$$\frac{\overline{F}_Z(z)}{z^{-b}} \underset{z\uparrow+\infty}{\rightarrow} +\infty. \quad (74)$$

Thus the tails of the distribution of Z are fatter than any power-law with exponent $b > 0$: they are very heavy tailed with tail index zero. As a result, such distributions have no moment of any arbitrary positive order. Thus model (70) for energy release in the subcritical regime is very heavy-tailed, with tail index zero; we shall call such models "very heavy tailed": fitting this distribution to a natural phenomenon would mean that there exists in Nature extreme situations with very heavy tails (and hence very special properties).

Let us make two remarks on this model, underlining its importance in our physical context.

Remark 5 As conventional wisdom suggests, the smaller a is, the heavier the tails for Z are, ranging from moderately heavy tails ($a > 1$) to very heavy tails ($a < 1$), through *Pareto*-like tails (71) with oscillating local exponent $s_Z(z) > 0$ in the critical situation when $a = 1$. In this model, the *Pareto*-like laws again appear as a critical phenomenon which suggests that such distributions are unlikely to be observed in Nature. Besides, the authors are unaware of any previous mention of such distributions in the literature.

Remark 6 As was noted in a straightforward way, for any value of $a > 0$, the energy variable Z is subexponential, *i.e.* presenting at least moderately heavy tails. In all these situations, the maximum $Z_{n:n} := \max(Z_1, ..., Z_n)$ of an n-sample $(Z_1, ..., Z_n)$ is tail equivalent to the sum $\sum_{m=1}^{n} Z_m$, [6], in the sense that

$$\frac{P(Z_{n:n} > x)}{P\left(\sum_{m=1}^{n} Z_m > x\right)} \underset{x\uparrow\infty}{\rightarrow} 1. \quad (75)$$

The tail of the maximum determines the tail of the sum.

If $a < 1$, hence with distributions with tail index zero, two stronger convergence results actually hold. They are

$$\frac{Z_{n:n}}{\sum_{m=1}^{n} Z_m} \rightarrow 1 \text{ (in probability)} \quad (76)$$

and even almost surely if and only if: $a < 1/2$ [29,24].

In this case, a single event explains (in probability or even almost surely) a cumulative event. The understanding of a cumulative energy goes through the one of its largest term.

5.1 Large deviation from the mean

From the law of large numbers, the empirical mean $\frac{1}{n}S_n := \frac{1}{n}\sum_{m=1}^{n} Y_m$ converges almost surely to the theoretical mean, say $\overline{m}_Y := EY$ of Y, known to exist as Y exhibits at most moderately heavy tails. Large deviation theory is concerned with the evaluation of the (small) probability

$$P\left(\frac{1}{n}S_n > y\right) \tag{77}$$

as y exceeds the mean $\overline{m}_Y$. More precisely, it is concerned with the rate at which this probability tends to zero, as a function of the sample size n. We shall distinguish two cases from the tail behavior of Y.

- $a \geq 1$. In this case the *Laplace* transform of $F_Y(y)$, say

  $$Z_Y(\beta) := \int_{y_0}^{\infty} \mathrm{e}^{-\beta y} \mathrm{d}F_Y(y) \tag{78}$$

  (with real β) is defined in a open neighborhood of $\beta = 0$. As a result, it is well-known that, with $y > \overline{m}_Y$

  $$n^{-1} \log P\left(\frac{1}{n}S_n > y\right) \underset{n\uparrow\infty}{\rightarrow} f(y) < 0. \tag{79}$$

  Here $f(y)$ the concave *Cramér-Chernoff* transform of the "free energy" $-\log Z_Y(\beta)$. The probability that the empirical observed mean deviates from the mean tends to zero exponentially fast as n grows, at rate $f(y)$.
- $a < 1$. The function $Z_Y(\beta)$ is no longer defined in an open neighborhood of $\beta = 0$, so that the previous result does *not* hold in this (subexponential) case. However, from the tail equivalence of the maximum and sum for subexponential distributions, one gets for large n

  $$P\left(\frac{1}{n}S_n > y\right) \sim P\left(\frac{1}{n}Y_{n:n} > y\right) = 1 - (F_Y(ny))^n.$$

  From (68), with $y > \overline{m}_Y$

  $$1 - (F_Y(ny))^n \sim n \exp - s_{V^-}(y_0 - ny)(ny - y_0)^a$$

  showing that (79) should be replaced by

  $$n^{-a} \log P\left(\frac{1}{n}S_n > y\right) \overset{n\uparrow\infty}{\rightarrow} -s_{V^+}(0)\, y^a < 0. \tag{80}$$

  This formula exhibits a slower decay to zero of $P\left(\frac{1}{n}S_n > y\right)$, due to the presence of moderately heavy tails as $a < 1$.

Note that the arithmetic mean for the observed sequence $(Y_1, ..., Y_n)$, say $\frac{1}{n}S_n$, corresponds to a *geometric* mean of the hidden energy records $(Z_1, ..., Z_n)$: $\prod_{m=1}^{n} Z_m^{1/n}$. The law of large numbers which states that $\frac{1}{n}S_n \rightarrow \overline{m}_Y$ almost surely obviously reads in terms of the energy records

$$\prod_{m=1}^{n} Z_m^{1/n} \underset{n\uparrow\infty}{\rightarrow} \exp \overline{m}_Y, \text{ almost surely.} \tag{81}$$

This observation and the large deviation results just mentioned show that there are large deviation results for the energy sequence itself, $(Z_m, m \geq 1)$ but not in terms of its empirical arithmetic mean (it simply could not converge as a result of heavy-tailedness of its distribution), rather of its empirical geometrical mean. In explicit form, (79, 80) read in terms of the geometrical mean of the energies

$$n^{-1} \log P\left(\prod_{m=1}^{n} Z_m^{1/n} > z\right) \underset{n\uparrow\infty}{\rightarrow} f(\log(z/z_0)) < 0 \quad \text{as } a \geq 1 \tag{82}$$

$$n^{-a} \log P\left(\prod_{m=1}^{n} Z_m^{1/n} > z\right) \underset{n\uparrow\infty}{\rightarrow} -s_{V^+}(0) \log(z/z_0)^a < 0 \quad \text{as } a < 1. \tag{83}$$

The fact that this *min-semistable* observable exhibit *at most* moderately heavy right tails and the logarithmic scale arguments suggest that when dealing with this sequence of *log-minsemistable* intensities, if one is to understand an average event, one should rather work either with the maximum of the sequence or through geometrical averaging since the standard arithmetic averaging of energies may be extremely ill-defined.

6 Discussion of main results

Let us summarize the main results of the preceding sections on max- and *log*-max semistable distributions.

We first recall the importance in Statistics and underline some properties of the *max-stable Fréchet-Weibull-Gumbel* distributions which are known to embody the only possible limiting distributions for centered and normalized maxima of *iid* random variables in this version of the Central Limit Theorem.

We then focus on a concept whose statistical insight is indeed deeper than the one of *stability*, namely the one of *max-semistability*, leading to a larger class of distributions. These can be defined as the fixed point of some functional equation (an invariance principle) for their probability distribution function which basically reflects the statistical *self-similarity* properties of their solutions; the analogy of this statistical point of view with the ones of Discrete Scale Invariance and Log-periodicity arising from Renormalization Group theory in the Physics' literature is underlined but not solved. We characterize the possible solutions to this functional equation, extending the three *max-stable* distributions in that their scale parameters are no longer constant, but rather allowed to vary in a log-periodic fashion. These distributions stand as appealing candidates to model *observed* random events in the Natural Sciences: indeed, from this formulation, the observed "global" random event interprets as the maximum of a *Poissonian* number of "micro-events" but with unknown intensity, location and scale parameters.

It turns out that the first two solutions of these *max-semistable* distributions may also be viewed as the exponentials of the third and may thus be considered as *log-max-semistable* as well. To complete the picture, we exhibit, the six possible types of *log-max-semistable* (and also of *log-min-semistable*) distributions, adding four statistical distributions to the two already mentioned from this class.

Observable are often modeled through the logarithms of other quantities with much more physical meaning, such as the "energy" or intensity of some underlying phenomenon: this means that *max-* or *min-semistable* variables are to be considered as the observable of some "hidden" physical phenomenon, which therefore proves itself *log min-*(or *max*)*-semistable*. Consequently, it is argued that *such* distributions should be ubiquitous in the modelling of random events measured in logarithmic scale: this observation motivates the full description of these distributions which is supplied.

We finally focus on a particular *log min-semistable* model which exhibits some sort of a "*Pareto*" critical behavior when the structure parameter crosses the value one: the logarithmic scale argument suggests that when dealing with a sequence of such intensities, if one is to understand an "average" event, one should rather work either with the maximum term of the sequence or use geometrical rather than arithmetical averaging.

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