Stochastic Models

Parameter Estimation for the Square-Root Diffusions: Ergodic and Nonergodic Cases

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Version of record first published: 06 Nov 2012.

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PARAMETER ESTIMATION FOR THE SQUARE-ROOT DIFFUSIONS: ERGODIC AND NONERGODIC CASES

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□ *This article deals with the problem of parameter estimation in the Cox-Ingersoll-Ross (CIR) model $(X_t)_{t \geq 0}$. This model is frequently used in finance for example as a model for computing the zero-coupon bound price or as a dynamic of the volatility in the Heston model. When the diffusion parameter is known, the maximum likelihood estimator (MLE) of the drift parameters involves the quantities : $\int_0^t X_s ds$ and $\int_0^t \frac{ds}{X_s}$. At first, we study the asymptotic behavior of these processes. This allows us to obtain various and original limit theorems on our estimators, with different rates and different types of limit distributions. Our results are obtained for both cases : ergodic and nonergodic diffusion. Numerical simulations were processed using an exact simulation algorithm.*

1. INTRODUCTION

Over the last few years, an interesting process emerged and became quite popular in finance, after Cox-Ingersoll-Ross (CIR) proposed it for modelling short-term interest rates.[4] It is also used for modelling stochastic volatility in the Heston model.[10] The CIR process $(X_t)_{t \geq 0}$, also known as the square root diffusion, is solution to the stochastic differential equation (SDE)

$$dX_t = (a - bX_t)dt + \sqrt{2\sigma|X_t|}dW_t, \qquad (1)$$

where $X_0 = x > 0$, $a > 0$, $b \in \mathbb{R}$, $\sigma > 0$ and $(W_t)_{t \geq 0}$ is a standard Brownian motion. Under the above assumption, based on the parameters that we will

suppose valid throughout the article, this SDE has a unique strong solution $(X_t)_{t \geq 0}$ (see Ikeda and Watanabe,[11] p. 221) and from the comparison theorem for one-dimensional diffusion process (see Revuz and Yor,[25] p. 394) we deduce that $X_t \geq 0$. In the particular case $b = 0$ and $\sigma = 2$, we recover the square of a a-dimensional Bessel process starting at x. Let us recall now some basic properties on the CIR model and let $\tau_0 :=$ $\inf\{t \geq 0 | X_t = 0\}$, with the convention $\inf \emptyset = \infty$. In the case $a \geq \sigma$, the process is strictly positive, τ_0 is infinite almost surely, otherwise it is non-negative, which means that it can reach the state 0. More precisely, for $a < \sigma$ and $b \geq 0$, τ_0 is finite almost surely and for $a < \sigma$ and $b < 0$, we have $\mathbb{P}_x(\tau_0 < \infty) \in]0, 1[$. Note that in the case $0 < a < \sigma$, when the process reaches the boundary, the state 0 is instantaneously reflecting (see, e.g., Göing-Jaeschke and Yor[8] or Lamberton and Lapeyre[19] for more details). From the ergodicity point of view, the CIR process is ergodic and its stationary distribution, say π, is a Gamma law with shape parameter a/σ and scale parameter σ/b, provided that $b > 0$. In this case, for all $h \in L^1(\pi)$, $\frac{1}{t} \int_0^t h(X_s) ds$ converges almost surely to $\int_{\mathbb{R}} h(x)\pi(dx)$.

During the last decades, several authors studied the problem of estimating parameters in the drift coefficient when a diffusion process was observed continuously; this corresponds to observe the path of the diffusion over an interval $[0, T]$, $T > 0$. This theory has been established mainly by Lipster and Shiryayev[20] and Kutoyants.[18] This approach is rather theoretical, since the real data are discrete time observations. However, if the error due to discretization is negligible, then the statistical results obtained for the continuous time model are valid for discrete time observations too. In the literature, most of the articles are concerned with ergodic diffusions and only few results can be found for the nonergodic case (see Ref.[18] section 3.1 of chapter 3 and references therein). In this last reference, many technics are proposed to construct estimators of the drift parameters. Furthermore, when the drift coefficient depends linearly on the parameters, one can hope to obtain a nice explicit formula of the maximum likelihood estimator (MLE).

Several papers have studied the MLE for the problem of estimating parameters in the CIR model. Fournié and Talay[7] have established the asymptotic normality in the subdomain of the two-dimensional parameter space $(a, b) \in]\sigma, \infty[\times]0, \infty[$. Afterward, Overbeck[22] considers the CIR model in the context of a continuous branching process with immigration rate $a > 0$ (CBI) and he gives more detailed results by including singular subdomains. By establishing all three properties—LAN (local asymptotic normality), LAMN (local asymptotic mixed normality) and LAQ (local asymptotic quadraticity)—he obtains consistency and asymptotic normality for the MLE for different submodels. Simultaneously, Overbeck and Ryden[23] examine the same problem from the discrete point of view. On the process itself, an important old reference for CBI is Kawazu and

Watanabe[15] (see also a recent work by Keller-Ressel and Mijatovic[16] and references their).

The aim of this article is to investigate the MLE of the drift parameters in the CIR model for a range of values (a, b, σ) covering ergodic and nonergodic situations. Roughly speaking, if we estimate one of the drift parameters and suppose known the other one, the MLE error has the form $M_t/\langle M \rangle_t$, where $(M_t)_{t \geq 0}$ is a Brownian martingale with quadratic variation $\langle M \rangle_t$. If $b > 0$ and $a > \sigma$, the asymptotic normality of the estimators is obtained using the classical martingale central limit theorem, as in Fournié and Talay.[7] Otherwise, this argument is no more valid, even in the special ergodic case $b > 0$ and $a \leq \sigma$, since $\int_{\mathbb{R}} (1/x) \pi(dx) = \infty$. To overcome this difficulty, we study the asymptotic behavior of the couple $(M_t, \langle M \rangle_t)$.

In the second section, as in our framework $\langle M \rangle_t$ is either $\int_0^t X_s ds$ or $\int_0^t \frac{ds}{X_s}$, we proceed by computing their Laplace transform. The first one is well known (see, e.g., Lamberton and Lapeyre,[19] p. 127). However for the second one, which is more subtle, we apply recent results of Craddock and Lennox,[5] who employ Lie symmetry methods to evaluate certain expectations for a large class of Itô diffusions. This allows us to obtain a precise description of the asymptotic of the $\int_0^t X_s ds$ (see Proposition 1 and 3) and $\int_0^t \frac{ds}{X_s}$ (see Proposition 2 and 4).

In the third section, we take advantage of this study to prove new original results on the asymptotic of the MLE that are not necessarily normal. The asymptotic theorem concerning the MLE, $\hat{b}_T$ (resp. $\hat{a}_T$), of b (resp. a) is obtained with different rates of convergence that are unusual in most cases : $\sqrt{T}$ for $b > 0$, T for $b = 0$ and $e^{-bT/2}$ for $b < 0$ (resp. $\sqrt{T}$ for $b > 0$ and $a > \sigma$, T for $b > 0$ and $a = \sigma$, $\sqrt{\log T}$ for $b = 0$ and $a > \sigma$ and $\log T$ for $b = 0$ and $a = \sigma$). In those different cases, the corresponding limit distributions are given by Theorem 1 for the MLE of b and Theorem 2 for the MLE of a.

Finally in the last section, we illustrate our asymptotic results using an exact simulation method. Indeed, to simulate the couple $(X_T, \int_0^T X_s ds)$ we use and perfect the method of Broadie and Kaya[3] based on an explicit evaluation of the conditional Laplace transform of $\int_0^T X_s ds$ given X_T. Concerning $(X_T, \int_0^T \frac{ds}{X_s})$, we first establish a new explicit formula of the conditional Laplace transform of $\int_0^T \frac{ds}{X_s}$ given X_T (see Theorem 3 and 4) and then we deduce an exact simulation method of the couple in the same manner as in Ref.[3]

2. THE ASYMPTOTIC BEHAVIOR OF $\int_0^t X_s ds$ AND $\int_0^t \frac{ds}{X_s}$

Let us recall that $(X_t)_{t \geq 0}$ denotes a CIR process solution to (1). It is relevant to consider separately the cases $b = 0$ and $b \neq 0$, since the process $(X_t)_{t \geq 0}$ behaves differently.

2.1. Case $b = 0$

In this section, we consider the Cox-Ingersoll-Ross CIR process $(X_t)_{t\geq 0}$ with $b = 0$. In this particular case, $(X_t)_{t\geq 0}$ satisfies the SDE

$$dX_t = a\,dt + \sqrt{2\sigma X_t}\,dW_t. \tag{2}$$

Note that for $\sigma = 2$, we recover the square of a a-dimensional Bessel process starting at x and denoted by BESQ_x^a. This process has been attracting considerably the attention of several studies (see Revuz and Yor[25]). The asymptotic behavior of $(X_t, \int_0^t X_s\,ds)$ and $\int_0^t \frac{ds}{X_s}$ are established by our next two propositions. Related results can be found in lemmas 4 and 5 of Overbeck.[22]

Proposition 1.   *Let $(X_t)_{t\geq 0}$ be a CIR process solution to (2), we have*

$$\left(\frac{X_t}{t}, \frac{1}{t^2}\int_0^t X_s\,ds\right) \xrightarrow{law} (R_1, I_1) \quad \text{as } t \text{ tends to infinity.}$$

where $(R_t)_{t\geq 0}$ is the CIR process starting from 0, solution to (2) and $I_t = \int_0^t R_s\,ds$.

Proposition 2.   *Under the above notations, we have*

1. $\mathbb{P}_x\left(\int_0^t \frac{ds}{X_s} < \infty\right) = 1$ *if and only if $a \geq \sigma$.*
2. *If $a > \sigma$ then $\frac{1}{\log t}\int_0^t \frac{ds}{X_s} \xrightarrow{\mathbb{P}} \frac{1}{a-\sigma}$ as t tends to infinity.*
3. *If $a = \sigma$ then $\frac{1}{(\log t)^2}\int_0^t \frac{ds}{X_s} \xrightarrow{law} \tau_1$ as t tends to infinity, where τ_1 is the hitting time associated with Brownian motion $\tau_1 := \inf\{t > 0 : W_t = \frac{1}{\sqrt{2\sigma}}\}$.*

In order to prove these propositions, we choose to compute the Laplace transform of the couple $(X_t, \int_0^t X_s\,ds)$ and $\int_0^t \frac{ds}{X_s}$. Here are the obtained results.

Lemma 1.   *We have*

• *For $\lambda \geq 0$ and $\mu \geq 0$,*

$$\mathbb{E}_x\left(e^{-\lambda X_t - \mu\int_0^t X_s\,ds}\right) = e^{-a\phi_{\lambda,\mu}(t)}e^{-x\psi_{\lambda,\mu}(t)},$$

where functions $\phi_{\lambda,\mu}$ and $\psi_{\lambda,\mu}$ are given by

$$\phi_{\lambda,\mu}(t) = \frac{1}{\sigma}\log\left(\frac{2\sigma\lambda}{\rho}\sinh(\rho t/2) + \cosh(\rho t/2)\right),$$

$$\psi_{\lambda,\mu}(t) = \frac{\lambda\cosh(\rho t/2) + \frac{2\mu}{\rho}\sinh(\rho t/2)}{\frac{2\sigma\lambda}{\rho}\sinh(\rho t/2) + \cosh(\rho t/2)}$$

and $\rho = 2\sqrt{\sigma\mu}$.

- *For $\mu > 0$,*

$$
\mathbb{E}_x\left(e^{-\mu \int_0^t \frac{ds}{X_s}}\right) = \frac{\Gamma(k + \frac{v}{2} + \frac{1}{2})}{\Gamma(v+1)} \left(\frac{x}{\sigma t}\right)^{\frac{v}{2} + \frac{1}{2} - k}
$$

$$
\exp\left(-\frac{x}{\sigma t}\right) {}_1F_1\left(k + \frac{v}{2} + \frac{1}{2}, v+1, \frac{x}{\sigma t}\right), \qquad (3)
$$

where $k = \frac{a}{2\sigma}$, $v = \frac{1}{\sigma}\sqrt{(a-\sigma)^2 + 4\mu\sigma}$ and ${}_1F_1$ is the confluent hypergeometric function defined by ${}_1F_1(u, v, z) = \sum_{n=0}^{\infty} \frac{u_n}{v_n} \frac{z^n}{n!}$, with $u_0 = v_0 = 1$, and for $n \geq 1$, $u_n = \prod_{k=0}^{n-1}(u+k)$ and $v_n = \prod_{k=0}^{n-1}(v+k)$.

Proof. Taking $b = 0$ in Proposition 2.5 of chapter 6 in Ref.[19] we deduce the first assertion. For the second one, we apply Theorem 5.10 in Ref.[5] to our process. We have just to be careful with the misprint in formula (5.24) of Ref.[5] More precisely, we have to replace $\sqrt{Axy}$, in the numerator of the first term in the right-hand side of this formula, by $\sqrt{Ax/y}$. Hence, for $a > 0$ and $\sigma > 0$, we obtain the so called fundamental solution of the PDE $u_t = \sigma x u_{xx} + a u_x - (\frac{\mu}{x} + \lambda x) u, \lambda > 0, \mu > 0$:

$$
p(t, x, y) = \frac{\sqrt{\sigma\lambda}}{\sigma \sinh(\sqrt{\sigma\lambda}t)} \left(\frac{y}{x}\right)^{k-1/2}
$$

$$
\exp\left(-\frac{\sqrt{\sigma\lambda}(x+y)}{\sigma \tanh(\sqrt{\sigma\lambda}t)}\right) I_v\left(\frac{2\sqrt{\sigma\lambda}\sqrt{xy}}{\sigma \sinh(\sqrt{\sigma\lambda}t)}\right), \qquad (4)
$$

where I_v is the modified Bessel function of the first kind. This yields the Laplace transform of the couple $\left(\int_0^t X_s ds, \int_0^t \frac{ds}{X_s}\right)$, since $\mathbb{E}_x(e^{-\lambda \int_0^t X_s ds - \mu \int_0^t \frac{ds}{X_s}}) = \int_0^\infty p(t, x, y) dy$. Evaluation of this integral is routine, see formula 2 of section 6.643 in Ref.[9] Therefore, we get

$$
\mathbb{E}_x\left(e^{-\lambda \int_0^t X_s ds - \mu \int_0^t \frac{ds}{X_s}}\right)
$$

$$
= \frac{\Gamma(k + \frac{v}{2} + \frac{1}{2})}{\Gamma(v+1)} \left(\frac{\sqrt{\sigma\lambda}x \coth(\sqrt{\sigma\lambda}t)}{\sigma}\right)^{-k} \exp\left(-\frac{\sqrt{\sigma\lambda}x}{\sigma}\coth(\sqrt{\sigma\lambda}t)\right)
$$

$$
\times \exp\left(\frac{\sqrt{\sigma\lambda}x}{2\sigma \sinh(\sqrt{\sigma\lambda}t)\cosh\left(\sqrt{\sigma\lambda}t\right)}\right)
$$

$$
\times M_{-k,\frac{v}{2}}\left(\frac{\sqrt{\sigma\lambda}x}{\sigma \sinh(\sqrt{\sigma\lambda}t)\cosh\left(\sqrt{\sigma\lambda}t\right)}\right), \qquad (5)
$$

where $M_{s,r}(z)$ is the Whittaker function of the first kind given by

$$M_{s,r}(z) = z^{r+\frac{1}{2}} e^{-\frac{z}{2}} {}_1F_1\left(r - s + \frac{1}{2}, 2r + 1, z\right). \tag{6}$$

See Ref.[9] for more details about those special functions. By inserting relation (6) in (5) we obtain

$$\mathbb{E}_x\left(e^{-\lambda \int_0^t X_s \, ds - \mu \int_0^t \frac{ds}{X_s}}\right)$$

$$= \frac{\Gamma(k + \frac{v}{2} + \frac{1}{2})}{\Gamma(v + 1)} \left(\cosh(\sqrt{\sigma\lambda}t)\right)^{-\frac{v}{2} - \frac{1}{2} - k} \left(\frac{\sqrt{\sigma\lambda}x}{\sigma \sinh(\sqrt{\sigma\lambda}t)}\right)^{\frac{v}{2} + \frac{1}{2} - k}$$

$$\times \exp\left(-\frac{\sqrt{\sigma\lambda}x}{\sigma} \coth(\sqrt{\sigma\lambda}t)\right)$$

$$\quad {}_1F_1\left(k + \frac{v}{2} + \frac{1}{2}, v + 1, \frac{\sqrt{\sigma\lambda}x}{\sigma \sinh(\sqrt{\sigma\lambda}t)\cosh(\sqrt{\sigma\lambda}t)}\right).$$

We complete the proof by letting λ tend to 0. $\qquad\square$

Proof of Proposition 1.   Under the notations of the above Lemma, it is easy to check that

$$\phi_{\frac{\lambda}{t}, \frac{\mu}{t^2}}(t) = -\frac{1}{\sigma} \log\left(\frac{\sqrt{\sigma\mu}}{\sigma\lambda \sinh(\sqrt{\sigma\mu}) + \sqrt{\sigma\mu} \cosh(\sqrt{\sigma\mu})}\right), \quad \lim_{t\to\infty} \psi_{\frac{\lambda}{t}, \frac{\mu}{t^2}}(t) = 0$$

and

$$\lim_{t\to\infty} \mathbb{E}_x\left(e^{-\lambda \frac{X_t}{t} - \frac{\mu}{t^2} \int_0^t X_s \, ds}\right) = \left(\frac{\sqrt{\sigma\mu}}{\sigma\lambda \sinh(\sqrt{\sigma\mu}) + \sqrt{\sigma\mu} \cosh(\sqrt{\sigma\mu})}\right)^{\frac{a}{\sigma}}.$$

Noting that the first assertion of Lemma 1 remains valid with $x = 0$ (see Ref.[19]), we deduce that the obtained limit is simply the Laplace transform of the CIR process starting from 0, solution to (2) with $t = 1$. This completes the proof. $\qquad\square$

Remark.   It is worth to note that Proposition 1 can be obtained by a scaling argument, but in order to standardize the technics used in this section we dropped this idea.

Proof of Proposition 2.   For $a \geq \sigma$, by Lemma 1, we have

$$\mathbb{P}_x\left(\int_0^t \frac{ds}{X_s} < \infty\right) = \lim_{\mu\to 0} \mathbb{E}_x\left(e^{-\mu \int_0^t \frac{ds}{X_s}}\right) = 1.$$

In the case $a < \sigma$, we have

$$\mathbb{P}_x \left(\int_0^t \frac{ds}{X_s} < \infty \right) = \frac{1}{\Gamma(2 - \frac{a}{\sigma})} \left(\frac{x}{\sigma t} \right)^{1 - \frac{a}{\sigma}} \exp\left(-\frac{x}{\sigma t} \right) {}_1F_1 \left(1, 2 - \frac{a}{\sigma}, \frac{x}{\sigma t} \right).$$

Thanks to the following formula (see section 9.211 in Ref.[9])

$$_1F_1 \left(r, s, z \right) = \frac{\Gamma(s)}{\Gamma(r)\Gamma(s - r)} z^{1-s} \int_0^z e^u u^{r-1} (z - u)^{s-r-1} \, du,$$

for $\mathrm{Re}(s) > \mathrm{Re}(r) > 0$, we obtain

$$\mathbb{P}_x \left(\int_0^t \frac{ds}{X_s} < \infty \right) = \frac{e^{-\frac{x}{\sigma t}}}{\Gamma(1 - \frac{a}{\sigma})} \int_0^{\frac{x}{\sigma t}} e^u \left(\frac{x}{\sigma t} - u \right)^{-\frac{a}{\sigma}} \, du.$$

After the change of variable, $v = \frac{x}{\sigma t} - u$, the last relation becomes

$$\mathbb{P}_x \left(\int_0^t \frac{ds}{X_s} < \infty \right) = \frac{1}{\Gamma(1 - \frac{a}{\sigma})} \int_0^{\frac{x}{\sigma t}} e^{-v} v^{-\frac{a}{\sigma}} \, dv < 1.$$

For the second and the third assertions, we consider a positive function $\gamma(t)$ increasing to $+\infty$ when $t \to +\infty$. Using standard evaluations, it is easy to prove that

$$\lim_{t \to +\infty} \mathbb{E}_x \left(e^{-\frac{\mu}{\gamma(t)^2} \int_0^t \frac{ds}{X_s}} \right) = \lim_{t \to +\infty} \exp\left(-\frac{1}{2\sigma} \left(\sigma - a + \sqrt{(a - \sigma)^2 + \frac{4\mu\sigma}{\gamma(t)^2}} \right) \log(t) \right).$$

• If $a > \sigma$, let ε denotes a function such that $\lim_{x \to 0} \varepsilon(x) = 0$, we have

$$\lim_{t \to +\infty} \mathbb{E}_x \left(e^{-\frac{\mu}{\gamma(t)^2} \int_0^t \frac{ds}{X_s}} \right)$$

$$= \lim_{t \to +\infty} \exp\left(-\left(\frac{\mu}{\gamma(t)^2 (a - \sigma)} + \frac{1}{\gamma(t)^2} \varepsilon \left(\frac{1}{\gamma(t)^2} \right) \log(t) \right) \right)$$

$$= \exp\left(-\frac{\mu}{a - \sigma} \right), \text{ by taking } \gamma(t)^2 = \log(t).$$

• If $a = \sigma$

$$\lim_{t \to +\infty} \mathbb{E}_x \left(e^{-\frac{\mu}{\gamma(t)^2} \int_0^t \frac{ds}{X_s}} \right) = \lim_{t \to +\infty} \exp\left(-\frac{1}{\sqrt{\sigma}} \frac{\sqrt{\mu}}{\gamma(t)} \log(t) \right)$$

$$= \exp\left(-\frac{\sqrt{\mu}}{\sqrt{\sigma}} \right), \text{ by taking } \gamma(t) = \log(t).$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We now turn to the case $b \neq 0$.

2.2. Case $b \neq 0$

Let us resume the general model of the CIR given by relation (1) with $b \neq 0$, namely

$$dX_t = (a - bX_t)dt + \sqrt{2\sigma X_t}dW_t, \tag{7}$$

where $X_0 = x > 0$, $a > 0$, $b \in \mathbb{R}^*$, $\sigma > 0$. Note that this process may be represented in terms of a square Bessel process through the relation $X_t = e^{-bt}Y(\frac{\sigma}{2b}(e^{bt} - 1))$, where Y denotes a $\mathrm{BESQ}_x^{\frac{2a}{\sigma}}$. This relation results from simple properties of square Bessel processes (see, e.g., Göing-Jaeschke and Yor[8] and Revuz and Yor[25]). We can now formulate the main results of this subsection.

Proposition 3. *Let $(X_t)_{t \geq 0}$ be a CIR process solution to (7), we have*

1. *If $b > 0$ then $\frac{1}{t}\int_0^t X_s ds \xrightarrow{\mathbb{P}} \frac{a}{b}$ as t tends to infinity.*
2. *If $b < 0$ then $(e^{bt}X_t, e^{bt}\int_0^t X_s ds) \xrightarrow{law} (R_{t_0}, t_0 R_{t_0})$, as t tends to infinity, where $t_0 = -1/b$ and $(R_t)_{t \geq 0}$ is the CIR process, starting from x, solution to (2).*

Proposition 4. *Under the above notations, we have*

1. *$\mathbb{P}_x(\int_0^t \frac{ds}{X_s} < \infty) = 1$ if and only if $a \geq \sigma$.*
2. *If $b > 0$ and $a > \sigma$ then $\frac{1}{t}\int_0^t \frac{ds}{X_s} \xrightarrow{\mathbb{P}} \frac{b}{a-\sigma}$ as t tends to infinity.*
3. *If $b > 0$ and $a = \sigma$ then $\frac{1}{t^2}\int_0^t \frac{ds}{X_s} \xrightarrow{law} \tau_2$ as t tends to infinity, where τ_2 is the hitting time associated with Brownian motion $\tau_2 := \inf\{t > 0 : W_t = \frac{b}{\sqrt{2\sigma}}\}$.*
4. *If $b < 0$ and $a \geq \sigma$ then $\int_0^t \frac{ds}{X_s} \xrightarrow{law} I_{t_0} := \int_0^{t_0} R_s ds$ as t tends to infinity, where $t_0 = -1/b$ and $(R_t)_{t \geq 0}$ is the CIR process, starting from x, solution to (2).*

Remark. When $a > \sigma$ and $b > 0$ the CIR process is ergodic and the stationary distribution is a Gamma law with shape a/σ and scale σ/b. Let $\xi \stackrel{law}{=} \Gamma(a/\sigma, \sigma/b)$, according to the ergodic theorem, $\frac{1}{t}\int_0^t X_s ds \xrightarrow{\mathbb{P}-p.s.} \mathbb{E}(\xi) = \frac{a}{b}$ and $\frac{1}{t}\int_0^t \frac{ds}{X_s} \xrightarrow{\mathbb{P}-p.s.} \mathbb{E}(\frac{1}{\xi}) = \frac{b}{a-\sigma}$ as t tends to infinity. In this case we recover the first assertion of Proposition 3 and the second assertion of Proposition 4.

In order to prove these propositions we need the following result.

Lemma 2. *We have,*

- *for $\lambda \geq 0$ and $\mu \geq 0$, the Laplace transform of $(X_t, \int_0^t X_s ds)$ is given by*

$$\mathbb{E}_x\left(e^{-\lambda X_t - \mu \int_0^t X_s ds}\right) = e^{-a\tilde{\phi}_{\lambda,\mu}(t)}e^{-x\tilde{\psi}_{\lambda,\mu}(t)}, \tag{8}$$

where

$$\tilde{\phi}_{\lambda,\mu}(t) = -\frac{1}{\sigma}\log\left(\frac{2\rho e^{t(b-\rho)/2}}{2\sigma\lambda(1-e^{-\rho t})+(\rho-b)e^{-\rho t}+(\rho+b)}\right)$$

and

$$\tilde{\psi}_{\lambda,\mu}(t) = \frac{\lambda((\rho+b)e^{-\rho t}+(\rho-b))+2\mu(1-e^{-\rho t})}{2\sigma\lambda(1-e^{-\rho t})+(\rho-b)e^{-\rho t}+(\rho+b)},$$

with $\rho = \sqrt{b^2+4\sigma\mu}$.

- *For $\mu > 0$, the Laplace transform of $\int_0^t \frac{ds}{X_s}$ is given by*

$$\mathbb{E}_x\left(e^{-\mu\int_0^t \frac{ds}{X_s}}\right) = \frac{\Gamma(k+\frac{v}{2}+\frac{1}{2})}{\Gamma(v+1)}\frac{1}{x^k\alpha^k}\beta^{\frac{v}{2}+\frac{1}{2}}$$

$$\times \exp\left(\frac{b}{2\sigma}\left[at-\frac{2x}{e^{bt}-1}\right]\right) {}_1F_1\left(k+\frac{v}{2}+\frac{1}{2},v+1,\beta\right), \quad (9)$$

where $k=\frac{a}{2\sigma}$, $\alpha=\frac{be^{bt}}{\sigma(e^{bt}-1)}$, $\beta=\frac{bx}{\sigma(e^{bt}-1)}$ and $v=\frac{1}{\sigma}\sqrt{(a-\sigma)^2+4\mu\sigma}$.

Remark. Note that the limit of the above Laplace transform, when b goes to 0, in formula (9) allows us to recover relation (3).

Proof. The first assertion is given by Proposition 2.5 of chapter 6 in Ref.[19] For the second one, we first apply Theorem 5.7 of Ref.[5] for $b \in \mathbb{R}$. We obtain the fundamental solution of the PDE $u_t = \sigma x u_{xx} + (a-bx)u_x - \frac{\mu}{x}u$, $\mu > 0$:

$$p(t,x,y)$$

$$= \frac{|b|}{2\sigma\sinh(|b|t/2)}\left(\frac{y}{x}\right)^{a/(2\sigma)-1/2}$$

$$\times \exp\left(\frac{b}{2\sigma}\left[at+(x-y)\right]-\frac{|b|(x+y)}{2\sigma\tanh(|b|t/2)}\right)I_v\left(\frac{|b|\sqrt{xy}}{\sigma\sinh(|b|t/2)}\right).$$
$$(10)$$

By the parity of hyperbolic functions, we omit the $|.|$ in the above formula. This yields the Laplace transform of $\int_0^t \frac{ds}{X_s}$, since $\mathbb{E}_x(e^{-\mu\int_0^t \frac{ds}{X_s}}) = \int_0^\infty p(t,x,y)dy$. In the same manner as in the proof of Lemma 1, formula 2 of section 6.643 in Ref.[9] gives us

$$\mathbb{E}_x\left(e^{-\mu\int_0^t \frac{ds}{X_s}}\right) = \frac{\Gamma(k+\frac{v}{2}+\frac{1}{2})}{\Gamma(v+1)}\frac{e^{\frac{\beta}{2}}}{x^k\alpha^k}\exp\left(\frac{b}{2\sigma}\left[at-\frac{2x}{e^{bt}-1}\right]\right)M_{-k,\frac{v}{2}}(\beta). \quad (11)$$

Finally, by inserting relation (6) in (11) we obtain the announced result.

$\square$

Remark.   In the above proof, relation (10) extends Corollary 5.8 of Ref.[5] established in the case $b > 0$, to the case $b \in \mathbb{R}$. It is worth to note that formula (5.20) in this Corollary remains valid for this extension, thanks to the parity of hyperbolic functions.

In the following proofs ε will denotes a function satisfying $\lim_{x \to 0} \varepsilon(x) = 0$, that can change from an evaluation to an other.

Proof of Proposition 3.   Let $\gamma(t)$ be a (non-random) positive function increasing to $+\infty$ when $t \to +\infty$, we take $\lambda = 0$ and replace μ by $\mu/\gamma(t)^2$ in relation (8). In the case $b > 0$, an easy computation shows that

$$\lim_{t \to +\infty} \mathbb{E}_x \left(e^{-\frac{\mu}{\gamma(t)^2} \int_0^t X_s ds} \right) = \lim_{t \to +\infty} \exp\left(-\frac{ab}{2\sigma} t \left(\sqrt{1 + \frac{4\mu\sigma}{b^2 \gamma(t)^2}} - 1 \right) \right)$$

$$= \exp\left(-\frac{ab}{2\sigma} \frac{t}{\gamma(t)^2} \left(\frac{2\mu\sigma}{b^2} + \varepsilon(\frac{1}{\gamma(t)^2}) \right) \right).$$

The first assertion follows by choosing $\gamma(t)^2 = t$.

Next, we study the case $b < 0$. According to the first assertion of Lemma 2, we have

$$\mathbb{E}_x \left(e^{-\lambda e^{bt} X_t - \mu e^{bt} \int_0^t X_s ds} \right) = e^{-a\tilde{\phi}_{\lambda e^{bt}, \mu e^{bt}}(t)} e^{-x\tilde{\psi}_{\lambda e^{bt}, \mu e^{bt}}(t)}$$

with $\rho = \sqrt{b^2 + 4\sigma\mu e^{bt}}$. As the Taylor's expansion of $\rho + b$ is equal to $-\frac{2\sigma\mu}{b} e^{bt} + e^{bt}\varepsilon(e^{bt})$, we deduce that $\lim_{t \to +\infty}(\rho + b)t = 0$ and $\lim_{t \to +\infty}(\rho + b) e^{\rho t} = \lim_{t \to +\infty}(\rho + b)e^{-bt} = -\frac{2\sigma\mu}{b}$. Hence, it's easy to check that

$$\lim_{t \to +\infty} \tilde{\phi}_{\lambda e^{bt}, \mu e^{bt}}(t) = -\frac{1}{\sigma} \log \left(\frac{-b}{\sigma\lambda - b - \frac{\sigma}{b}\mu} \right) \quad \text{and}$$

$$\lim_{t \to +\infty} \tilde{\psi}_{\lambda e^{bt}, \mu e^{bt}}(t) = \frac{-b\lambda + \mu}{\sigma\lambda - b - \frac{\sigma}{b}\mu}.$$

Therefore

$$\lim_{t \to +\infty} \mathbb{E}_x \left(e^{-\lambda e^{bt} X_t - \mu e^{bt} \int_0^t X_s ds} \right) = \left(\frac{-b}{\sigma\lambda - b - \frac{\sigma}{b}\mu} \right)^{a/\sigma} \exp\left(-x \frac{-b\lambda + \mu}{\sigma\lambda - b - \frac{\sigma}{b}\mu} \right).$$

On the other hand, using the first assertion of Lemma 3, we identify the Laplace transform of the announced couple limit $(R_{t_0}, t_0 R_{t_0})$, which completes the proof.   □

Remark. For $b < 0$, we can also give this representation $e^{bt} \int_0^t X_s ds \xrightarrow{law} L := L1 + L2$ as t tends to infinity where $L1$ and $L2$ are two independent random variables, $L1$ has a gamma distribution and $L2$ has a compound Poisson distribution. More precisely $L_1 \stackrel{law}{=} \Gamma(a/\sigma, \sigma/b^2)$ and $L_2 \stackrel{law}{=} \sum_{k=1}^N X_k$ where $N \stackrel{law}{=} \mathscr{P}(xb/\sigma)$, for all $k \geq 1$, $X_k \stackrel{law}{=} \mathscr{E}(\sigma/b^2)$ and $(N, X_1, \ldots, X_n, \ldots)$ are mutually independent.

Proof of Proposition 4. Note that, the Laplace transform of $\int_0^t \frac{ds}{X_s}$ converges to $\mathbb{P}_x(\int_0^t \frac{ds}{X_s} < \infty)$ as $\mu \to 0$. If $a \geq \sigma$, using standard evaluations, it is easy to prove that

$$\mathbb{P}_x\left(\int_0^t \frac{ds}{X_s} < \infty\right) = \left(\frac{\beta}{x\alpha}\right)^{\frac{a}{2\sigma}} \exp\left(\frac{b}{2\sigma}\left[at - \frac{2x}{e^{bt} - 1}\right] + \beta\right) = 1.$$

In the other case, $a < \sigma$, we have

$$\mathbb{P}_x\left(\int_0^t \frac{ds}{X_s} < \infty\right)$$
$$= \frac{1}{\Gamma(2 - \frac{a}{\sigma})} \frac{\beta^{1-\frac{a}{2\sigma}}}{(x\alpha)^{\frac{a}{2\sigma}}} \exp\left(\frac{b}{2\sigma}\left[at - \frac{2x}{e^{bt} - 1}\right]\right) {}_1F_1\left(1, 2 - \frac{a}{\sigma}, \beta\right)$$
$$= \frac{\beta^{1-\frac{a}{\sigma}}}{\Gamma(2 - \frac{a}{\sigma})} e^{-\beta} {}_1F_1\left(1, 2 - \frac{a}{\sigma}, \beta\right).$$

Thanks to the following formula (see section 9.211 in Ref.[9])

$${}_1F_1(r, s, z) = \frac{\Gamma(s)}{\Gamma(r)\Gamma(s - r)} z^{1-s} \int_0^z e^u u^{r-1} (z - u)^{s-r-1} du,$$

for $\mathrm{Re}(s) > \mathrm{Re}(r) > 0$, we obtain

$$\mathbb{P}_x\left(\int_0^t \frac{ds}{X_s} < \infty\right) = \frac{e^{-\beta}}{\Gamma(1 - \frac{a}{\sigma})} \int_0^\beta e^u (\beta - u)^{-\frac{a}{\sigma}} du.$$

After the change of variable, $v = \beta - u$, the last relation becomes

$$\mathbb{P}_x\left(\int_0^t \frac{ds}{X_s} < \infty\right) = \frac{1}{\Gamma(1 - \frac{a}{\sigma})} \int_0^\beta e^{-v} v^{-\frac{a}{\sigma}} dv < 1.$$

Now our task is to study the asymptotic behavior in distribution of the quantity $\int_0^t \frac{ds}{X_s}$ when $b > 0$ and $a \geq \sigma$. Let $\gamma(t)$ be a (non-random) positive function increasing to $+\infty$ when $t \to +\infty$. If we replace μ by $\mu/\gamma(t)^2$

in relation (9), since $\log(\beta) = \log(bx/\sigma) - bt - \log(1 - e^{-bt})$,

$$\lim_{t \to +\infty} \mathbb{E}_x \left(e^{-\frac{\mu}{\gamma(t)^2} \int_0^t \frac{ds}{X_s}} \right)$$

$$= \lim_{t \to +\infty} \frac{1}{x^k \alpha^k} \exp\left(\frac{ab}{2\sigma} t - \left[\frac{1}{2\sigma} \sqrt{(a-\sigma)^2 + \frac{4\mu\sigma}{\gamma(t)^2}} + \frac{1}{2} \right] \log(\beta) \right)$$

$$= \lim_{t \to +\infty} \exp\left(\frac{ab}{2\sigma} t - \left[\frac{1}{2\sigma} \sqrt{(a-\sigma)^2 + \frac{4\mu\sigma}{\gamma(t)^2}} + \frac{1}{2} \right] bt \right).$$

- For $a > \sigma$, we have

$$\lim_{t \to +\infty} \mathbb{E}_x \left(e^{-\frac{\mu}{\gamma(t)^2} \int_0^t \frac{ds}{X_s}} \right)$$

$$= \lim_{t \to +\infty} \exp\left(\frac{ab}{2\sigma} t - \left[\frac{a-\sigma}{2\sigma} \sqrt{1 + \frac{4\mu\sigma}{\gamma(t)^2(a-\sigma)^2}} + \frac{1}{2} \right] bt \right)$$

$$= \lim_{t \to +\infty} \exp\left(\frac{\mu bt}{\gamma(t)^2(a-\sigma)} + \frac{t}{\gamma(t)^2} \varepsilon\left(\frac{1}{\gamma(t)^2} \right) \right).$$

Taking $\gamma(t)^2 = t$, we deduce that $\frac{1}{t} \int_0^t \frac{ds}{X_s}$ converges in distribution and of course in probability to the constant $b/(a - \sigma)$.

- For $a = \sigma$, we have

$$\lim_{t \to +\infty} \mathbb{E}_x \left(e^{-\frac{\mu}{\gamma(t)^2} \int_0^t \frac{ds}{X_s}} \right) = \lim_{t \to +\infty} \exp\left(-\frac{bt}{\sqrt{\sigma}\gamma(t)} \sqrt{\mu} \right).$$

Taking now $\gamma(t) = t$, we deduce that $\frac{1}{t^2} \int_0^t \frac{ds}{X_s}$ converges in distribution to τ_2.

It remains now to prove the last assertion. For $b < 0$, when t goes to infinity in relation (9), since α converges to 0 and β to $-bx/\sigma$, we have

$$\lim_{t \to +\infty} \mathbb{E}_x \left(e^{-\mu \int_0^t \frac{ds}{X_s}} \right)$$

$$= \frac{\Gamma(k + \frac{v}{2} + \frac{1}{2})}{\Gamma(v+1)} \left(\frac{-bx}{\sigma} \right)^{\frac{v}{2}+\frac{1}{2}} \frac{1}{x^k} \, {}_1F_1\left(k + \frac{v}{2} + \frac{1}{2}, v+1, \frac{-bx}{\sigma} \right)$$

$$\times \lim_{t \to +\infty} \exp\left(\frac{b}{2\sigma} [at + 2x] - k \log \alpha \right)$$

$$= \frac{\Gamma(k + \frac{v}{2} + \frac{1}{2})}{\Gamma(v+1)} \left(\frac{-bx}{\sigma} \right)^{\frac{v}{2}+\frac{1}{2}-k} \exp\left(\frac{bx}{\sigma} \right) {}_1F_1\left(k + \frac{v}{2} + \frac{1}{2}, v+1, \frac{-bx}{\sigma} \right).$$

Finally, we conclude by identifying the limit distribution with relation (3) in Lemma 1. □

3. STATISTICAL INFERENCE OF THE CIR MODEL

Let us first recall some basic notions on the construction of the maximum likelihood estimator (MLE). Suppose that the one dimensional diffusion process $(X_t)_{t \geq 0}$ satisfies

$$dX_t = b(\theta, X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x_0,$$

where the parameter $\theta \in \Theta \subset \mathbb{R}^p$, $p \geq 1$, is to be estimated. The coefficients b and σ are two functions satisfying conditions that guarantee the existence and uniqueness of the SDE for each $\theta \in \Theta$. We denote by $\mathbb{P}_\theta$ the probability measure induced by the solution of the equation on the canonical space $C(\mathbb{R}_+, \mathbb{R})$ with the natural filtration $\mathscr{F}_t := \sigma(W_s, s \leq t)$, and let $\mathbb{P}_{\theta,t} := \mathbb{P}_\theta | \mathscr{F}_t$ be the restriction of $\mathbb{P}_\theta$ to $\mathscr{F}_t$. If the integrals in the next formula below make sense then the measures $\mathbb{P}_{\theta,t}$ and $\mathbb{P}_{\theta_0,t}$, for any $\theta, \theta_0 \in \Theta$, $t > 0$, are equivalent (see Jacod,[12] Jacod and Shiryaev[13] (e.g., Ch. III.3) and Lipster and Shirayev[20] (e.g., Ch. 7) and we are able to introduce the so called likelihood ratio

$$
\begin{aligned}
L_t^{\theta,\theta_0} &:= \frac{d\mathbb{P}_{\theta,t}}{d\mathbb{P}_{\theta_0,t}} \\
&= \exp\left\{ \int_0^t \frac{b(\theta, X_s) - b(\theta_0, X_s)}{\sigma^2(X_s)} dX_s - \frac{1}{2} \int_0^t \frac{b^2(\theta, X_s) - b^2(\theta_0, X_s)}{\sigma^2(X_s)} ds \right\}.
\end{aligned}
$$

(12)

The process $\left(L_t^{\theta,\theta_0} \right)_{t \geq 0}$ is an $\mathscr{F}_t$−martingale.

In the present section, we observe the process $X^T = (X_t)_{0 \leq t \leq T}$ as a parametric model solution to equation (1), namely

$$dX_t = (a - bX_t)dt + \sqrt{2\sigma X_t}dW_t, \tag{13}$$

where $X_0 = x > 0$, $a > 0$, $b \in \mathbb{R}$, $\sigma > 0$. The unknown parameter, say θ, is involved only in the drift part of the diffusion and we consider the two cases:

- under fixed value of $a > 0$, we consider laws of processes (13) parameterized by $\theta = b \in \mathbb{R}$,
- under fixed value of $b \in \mathbb{R}$, we consider laws of the processes parameterized by $\theta = a \in [\sigma, \infty[$.

This study includes both ergodic ($b > 0$ and $a \geq \sigma$) and nonergodic cases.

3.1. Parameter Estimation $\theta = b$

Fix $a > 0$ and consider the submodel $\{\mathbb{P}_{\theta,t} : \theta = b \in \mathbb{R}\}$ for laws of processes (13) parameterized by $b \in \mathbb{R}$. In this case, we do have local equivalence of probability measures. The appropriate likelihood ratio (12), evaluated at time T with $\theta_0 = 0$, is well defined and is given by

$$L_T(b) := L_T^{\theta,\theta_0} = \exp\left\{ \frac{b}{2\sigma}(x - X_T) + \frac{1}{4\sigma}\int_0^T (2ab - b^2 X_s)ds \right\}.$$

The MLE $\hat{b}_T$ of b maximizes $L_T(b)$, then

$$\hat{b}_T = \frac{aT + x - X_T}{\int_0^T X_s ds}. \tag{14}$$

Hence, the error is given by

$$\hat{b}_T - b = -\sqrt{2\sigma}\frac{\int_0^T \sqrt{X_s}dW_s}{\int_0^T X_s ds}.$$

As mentioned in the introduction, the above error is obviously of the form $M_t/\langle M \rangle_t$, where $(M_t)_{t \geq 0}$ is a Brownian martingale with quadratic variation $\langle M \rangle_t$. If this quadratic variation, correctly normalized, converges in probability then the classical martingale central limit theorem can be applied. Otherwise, the study of the couple $(X_T, \int_0^T X_s)$ will be helpful to investigate the limit law of the error. The asymptotic behavior of $\hat{b}_T - b$ can be summarized as follows.

Theorem 1. *The MLE of b satisfies*

1. *Case $b > 0$:* $\mathscr{L}_b\left\{ \sqrt{T}(\hat{b}_T - b) \right\} \Longrightarrow \mathscr{N}\left(0, 2\sigma\frac{b}{a}\right).$

2. *Case $b = 0$:* $\mathscr{L}_b\left\{ T(\hat{b}_T - b) \right\} \Longrightarrow \frac{a - R_1}{I_1}$, *where (R_t) is the CIR process, starting from 0, solution to (2) and $I_t = \int_0^t R_s ds$.*

3. *Case $b < 0$:* $\mathscr{L}_b\left\{ e^{-bT/2}(\hat{b}_T - b) \right\} \Longrightarrow \frac{G}{R}$, *where (G, R) is a couple of random variable characterized with its joint moment generating-Laplace transform. For $\lambda \in \mathbb{R}$ and $\mu \geq 0$,*

$$\mathbb{E}\left(e^{\lambda G - \mu R}\right) = \left(\frac{b}{\mu\sigma/b + b}\right)^{\frac{a}{\sigma}} \exp\left(x\frac{\sigma\lambda^2/b + \mu}{\mu\sigma/b + b}\right).$$

Therefore G and R are correlated, G is normal and R has the same distribution as $t_0 R_{t_0}$, $t_0 = -1/b$, where $(R_t)_{t \geq 0}$ is the CIR process, starting from x, solution to (2).

Proof. In the case $b > 0$, by Proposition 3 and the central limit theorem given by Y.A. Kutoyants (see Theorem 1.19 in Ref.[18]) we have

$$\mathbb{P}_b - \lim_{T \to \infty} \frac{1}{T} \int_0^T X_s ds = \frac{a}{b}, \quad \mathscr{L}_b \left\{ \frac{1}{\sqrt{T}} \int_0^T \sqrt{X_s} dW_s \right\} \Longrightarrow \mathscr{N} \left(0, \frac{a}{b} \right).$$

Therefore we obtain the first assertion. In the second case $b = 0$, by Proposition 1 we have

$$\mathscr{L}_0 \left\{ \frac{X_T}{T}, \frac{1}{T^2} \int_0^T X_s ds \right\} \Longrightarrow (R_1, I_1).$$

Hence

$$\mathscr{L}_0 \left\{ T(\hat{b}_T - b) \right\} = \mathscr{L}_0 \left\{ \frac{a + \frac{x}{T} - \frac{X_T}{T}}{\frac{1}{T^2} \int_0^T X_s ds} \right\} \Longrightarrow \frac{a - R_1}{I_1}.$$

For the last case $b < 0$, by Proposition 3 we have

$$\mathscr{L}_b \left\{ e^{bT} X_T, e^{bT} \int_0^T X_s ds \right\} \Longrightarrow \left(\tilde{R}_{t_0}, t_0 \tilde{R}_{t_0} \right),$$

where $t_0 = -1/b$ and $(\tilde{R}_t)_{t \geq 0}$ is the CIR process, starting from x, solution to (2). It follows that

$$\mathscr{L}_b \left\{ \hat{b}_T - b \right\} = \mathscr{L}_b \left\{ \frac{e^{bT}(aT + x - X_T - b \int_0^T X_s ds)}{e^{bT} \int_0^T X_s ds} \right\} \Longrightarrow 0$$

and also in probability, which proves the consistency of $\hat{b}_T$. Now, in order to study the asymptotic behavior of $\hat{b}_T - b$, we introduce the joint moment generating-Laplace transform of the renormalized couple $\left(e^{bT/2} \left(-X_T - b \int_0^T X_s ds \right), e^{bT} \int_0^T X_s ds \right)$, namely for λ in the neighborhood of the origin and $\mu \geq 0$ we consider

$$\mathbb{E}_x \left(e^{-\lambda e^{bT/2}(X_T + b \int_0^T X_s ds) - \mu e^{bT} \int_0^T X_s ds} \right) = \mathbb{E}_x \left(e^{-\lambda e^{bT/2} X_T - (\lambda e^{bT/2} b + \mu e^{bT}) \int_0^T X_s ds} \right).$$

By noting that the result of the first assertion in Lemma 2 remains valid for small values of λ, the above quantity becomes equal to $e^{-a \bar{\phi}_{\lambda, \mu}(T)} e^{-x \bar{\psi}_{\lambda, \mu}(T)}$ where

$$\bar{\phi}_{\lambda, \mu}(T) = -\frac{1}{\sigma} \log \left(\frac{2 \rho e^{T(b + \rho)/2}}{2 \sigma \lambda (e^{(\rho + \frac{b}{2})T} - e^{\frac{b}{2}T}) + (\rho - b) + (\rho + b) e^{\rho T}} \right)$$

and

$$\bar{\psi}_{\lambda,\mu}(T) = \frac{\lambda e^{bT/2}((\rho + b) + (\rho - b)e^{\rho T}) + 2(\lambda e^{bT/2}b + \mu e^{bT})(e^{\rho T} - 1)}{2\sigma\lambda(e^{(\rho + \frac{b}{2})T} - e^{\frac{b}{2}T}) + (\rho - b) + (\rho + b)e^{\rho T}},$$

with $\rho = \sqrt{b^2 + 4\sigma(\lambda e^{bT/2}b + \mu e^{bT})}$. Since ρ goes to $-b$ and $(\rho + b)T$ goes to zero as T tends to infinity, we have

$$\lim_{T\to\infty} \bar{\phi}_{\lambda,\mu}(T) = -\frac{1}{\sigma}\lim_{T\to\infty} \log\left(\frac{-2b}{(2\sigma\lambda e^{\frac{b}{2}T} + \rho + b)e^{\rho T} - 2b}\right)$$

and

$$\lim_{T\to\infty} \bar{\psi}_{\lambda,\mu}(T) = \lim_{T\to\infty} \frac{\lambda e^{-bT/2}(\rho + b) + 2\mu}{(2\sigma\lambda e^{\frac{b}{2}T} + \rho + b)e^{\rho T} - 2b}.$$

A Taylor's expansion gives $\lim_{T\to\infty}(2\sigma\lambda e^{\frac{b}{2}T} + \rho + b)e^{\rho T} = -2\sigma\mu/b$ and $\lim_{T\to\infty} e^{-bT/2}(\rho + b) = 2\sigma\lambda/b$. This completes the proof. $\qquad\square$

3.2. Parameter Estimation $\theta = a$

Fix $b \in \mathbb{R}$ and consider the submodel $\{\mathbb{P}_{\theta,t} : \theta = a \in [\sigma, \infty[\}$ for laws of processes (13) parameterized by $a \in [\sigma, \infty[$. In this case, thanks to the first assertion of both propositions 2 and 4 we have, for some reference point $a_0 \in [\sigma, \infty[$, $\mathbb{P}_{a_0}(\int_0^T \frac{ds}{X_s} < \infty) = 1$ and $\mathbb{P}_a(\int_0^T \frac{ds}{X_s} < \infty) = 1$. Hence, we have local equivalence of probability measures on the restricted parameter space $[\sigma, \infty[$. According to relation (12), the appropriate likelihood ratio, evaluated at time T with $\theta_0 = a_0 \in [\sigma, \infty[$, makes sense and is given by

$$L_T(a) := L_T^{\theta,\theta_0} = \exp\left\{\frac{a - a_0}{2\sigma}\int_0^T \frac{dX_s}{X_s} - \frac{1}{4\sigma}\int_0^T \frac{a^2 - 2abX_s - a_0^2 + 2a_0bX_s}{X_s}ds\right\}.$$

The MLE $\hat{a}_T$ of a maximizes $L_T(a)$, then

$$\hat{a}_T = \frac{bT + \int_0^T \frac{dX_s}{X_s}}{\int_0^T \frac{ds}{X_s}}.$$

Hence, the error is given by

$$\hat{a}_T - a = \sqrt{2\sigma}\frac{\int_0^T \frac{dW_s}{\sqrt{X_s}}}{\int_0^T \frac{ds}{X_s}}.$$

As explained in the last subsection, we can apply the classical martingale central limit theorem when $\int_0^T \frac{ds}{X_s}$, correctly normalized, converges in probability. Otherwise, using Itô's formula, we rewrite the MLE as follows

$$\hat{a}_T = \frac{\log X_T - \log x + bT + \sigma \int_0^T \frac{ds}{X_s}}{\int_0^T \frac{ds}{X_s}} \tag{15}$$

and now we study the couple $(\log(X_T), \int_0^T \frac{ds}{X_s})$ in order to obtain a limit law for the error. The asymptotic behavior of $\hat{a}_T - a$ can be summarized as follows.

Theorem 2. *The MLE of a is well defined for $a \geq \sigma$ and satisfies*

1. *Case $b = 0$ and $a > \sigma$: $\mathscr{L}_a \left\{ \sqrt{\log T}(\hat{a}_T - a) \right\} \Longrightarrow \mathscr{N}(0, 2\sigma(a - \sigma))$.*
2. *Case $b > 0$ and $a > \sigma$: $\mathscr{L}_a \left\{ \sqrt{T}(\hat{a}_T - a) \right\} \Longrightarrow \mathscr{N}\left(0, \frac{2\sigma(a-\sigma)}{b}\right)$.*
3. *Case $b = 0$ and $a = \sigma$: $\mathscr{L}_a \left\{ (\log T)(\hat{a}_T - a) \right\} \Longrightarrow \frac{1}{\tau_1}$, where τ_1 is the hitting time associated with Brownian motion $\tau_1 = \inf\left\{ t > 0 : W_t = \frac{1}{\sqrt{2\sigma}} \right\}$.*
4. *Case $b > 0$ and $a = \sigma$: $\mathscr{L}_a \left\{ T(\hat{a}_T - a) \right\} \Longrightarrow \frac{b}{\tau_2}$, where τ_2 is the hitting time associated with Brownian motion $\tau_2 = \inf\left\{ t > 0 : W_t = \frac{b}{\sqrt{2\sigma}} \right\}$.*
5. *Case $b < 0$ and $a \geq \sigma$: the MLE estimator $\hat{a}_T$ is not consistent.*

Proof. At first, from Proposition 2 and 4, we have $\mathbb{P}_a(\int_0^T \frac{ds}{X_s} < \infty) = 1$ for $a \geq \sigma$.

In the case $b = 0$ and $a > \sigma$, by Proposition 2 and the central limit theorem, we have

$$\mathbb{P}_a - \lim_{T \to \infty} \frac{1}{\log T} \int_0^T \frac{ds}{X_s} = \frac{1}{a - \sigma}, \quad \mathscr{L}_a \left\{ \frac{1}{\sqrt{\log T}} \int_0^T \frac{dW_s}{\sqrt{X_s}} \right\} \Longrightarrow \mathscr{N}\left(0, \frac{1}{a - \sigma}\right).$$

This establishes the first assertion. The second one is obtained in the same manner using Proposition 4.

For the third case $b = 0$ and $a = \sigma$, relation (15) yields

$$\hat{a}_T - a = \frac{\log X_T - \log x}{\int_0^T \frac{ds}{X_s}}.$$

The task is now to find the asymptotic behavior of $\log X_T$. By a scaling argument the process $(X_{2t/a})$ has the same distribution as a bidimensional square Bessel process, starting from x, BESQ_x^2. It follows that

$$X_{2T/a} \overset{law}{=} \|B_T + x\|^2 \overset{law}{=} T\|B_1 + x/\sqrt{T}\|^2,$$

where $(B_t)_{t \geq 0}$ denotes a standard bidimensional Brownian motion. Hence, $\log X_T / \log T$ converges in law to one and consequently in probability. This gives us the announced result.

For the case $b > 0$ and $a = \sigma$, as in the above case we can rewrite the error

$$\hat{a}_T - a = \frac{\log X_T - \log x + bT}{\int_0^T \frac{ds}{X_s}}.$$

Now, the CIR process (X_t) can be represented in terms of a BESQ_x^2 as follows

$$X_T \overset{law}{=} e^{-bT} \text{BESQ}_x^2 \left(\frac{a}{2b} (e^{bT} - 1) \right).$$

Since $\log \text{BESQ}_x^2 (T) / \log T$ converges in distribution to one, we deduce that $(\log X_T - \log x + bT)/T$ converges in distribution to b and consequently in probability. This gives our claim.

The only point remaining concerns the last case, $b < 0$ and $a \geq \sigma$. On the one hand, according to the above representation in law of the CIR process as a time changed BESQ_x^2, we have the convergence in distribution of $\log X_T + bT$ to the logarithm of $\text{BESQ}_x^2(-a/2b)$. On the other hand, since $(\int_0^t \frac{ds}{X_s})_{t \geq 0}$ is an increasing process, we deduce the almost surely convergence in the last assertion of Proposition 4. This completes the proof. □

Remark. Note that the last assertion of the above theorem is consistent with Theorem 2 v) of Overbeck,[22] where he proves that in the case $a > 0$ and $b < 0$ there is no consistent estimator of a.

4. NUMERICAL SIMULATIONS

Our aim in this section is to illustrate and test the practical behavior of the estimators errors stated above. For this purpose, we need to generate at time $T > 0$, the CIR X_T, the so called Lévy area $\int_0^T X_s ds$ and $\int_0^T \frac{ds}{X_s}$.

One way to do that is to use numerical schemes solving the SDE, like the famous Euler scheme. However, any discretization scheme introduces bias into the simulation results; an extensive discussion on this subject is given by Kloeden and Platen.[17] Nevertheless, it is important to note that the discretization of the the CIR process present some troubles because of the square root in the diffusion coefficient. Several papers deal with this problem, see for example Alfonsi,[1] Berkaoui, Bossy and Diop,[2] and Deelstra and Delbaen.[6] However, the proposed schemes do not cover the nonergodic case in all its generality.

In the other hand, it is well known that the CIR X_T is a non-central chi-squared random variable that can be simulated exactly. Therefore, Broadie and Kaya[3] propose an exact simulation of $(X_T, \int_0^T X_s ds)$ after they compute the conditional Laplace transform of $\int_0^T X_s ds$ given X_T. In our context, we proceed analogously for the simulation of $\left(X_T, \int_0^T \frac{ds}{X_s}\right)$. For this aim, we have to explicit also the conditional Laplace transform of $\int_0^T \frac{ds}{X_s}$ given X_T. This result is established in the subsection below to be used then in simulation.

4.1. On The Conditional Law of $\int_0^t X_s ds$ and $\int_0^t \frac{ds}{X_s}$ Given X_t

First, we give the conditional Laplace transform for the couple $\left(\int_0^t X_s ds, \int_0^t \frac{ds}{X_s}\right)$ given X_t, $t > 0$, where $(X_t)_{t \geq 0}$ denotes a BESQ$_x^a$ process, starting from x, solution to (2) with $\sigma = 2$, namely $dX_t = a dt + 2\sqrt{X_t} dW_t$.

Theorem 3. *For the square Bessel process introduced above, we have*

$$
\mathbb{E}_x \left(e^{-\lambda \int_0^t X_s ds - \mu \int_0^t \frac{ds}{X_s}} | X_t = y \right)
$$

$$
= \frac{\sqrt{2\lambda} t}{\sinh(\sqrt{2\lambda} t)} \exp\left(\frac{x+y}{2t} \left[1 - \sqrt{2\lambda} t \coth(\sqrt{2\lambda} t) \right] \right)
$$

$$
\times \frac{I_\gamma \left(\sqrt{2\lambda xy} / \sinh(\sqrt{2\lambda} t) \right)}{I_\nu \left(\sqrt{xy}/t \right)} \tag{16}
$$

for all $\lambda \geq 0$ and $\mu > 0$, with $\gamma = \sqrt{(a-2)^2 + 8\mu}/2$ and $\nu = a/2 - 1$.

Proof. On the one hand, by taking $\sigma = 2$ in equation (4), we deduce that the fundamental solution to the PDE $u_t = 2x u_{xx} + a u_x - (\lambda x + \frac{\mu}{x}) u$, satisfying the relation $\mathbb{E}_x \left(e^{-\lambda \int_0^t X_s ds - \mu \int_0^t \frac{ds}{X_s}} \right) = \int_0^{+\infty} p(t, x, y) dy$, is given by

$$
p(t, x, y) = \frac{\sqrt{2\lambda}}{2 \sinh(\sqrt{2\lambda} t)} \left(\frac{y}{x} \right)^{\nu/2} \exp\left(-\frac{\sqrt{2\lambda}}{2} (x+y) \coth(\sqrt{2\lambda} t) \right)
$$

$$
I_\gamma \left(\sqrt{2\lambda xy} / \sinh(\sqrt{2\lambda} t) \right).
$$

On the other hand, by conditioning we rewrite

$$
\mathbb{E}_x \left(e^{-\lambda \int_0^t X_s ds - \mu \int_0^t \frac{ds}{X_s}} \right) = \int_0^\infty \mathbb{E}_x \left(e^{-\lambda \int_0^t X_s ds - \mu \int_0^t \frac{ds}{X_s}} | X_t = y \right) p_{X_t}(y) dy,
$$

where $p_{X_t} = \frac{1}{2t} (\frac{y}{x})^{\nu/2} \exp(\frac{x+y}{2t}) I_\nu(\sqrt{xy}/t)$ is the density of the Bessel process starting from x. The result follows by identification. $\square$

Remarks.   Here we comment the above result, in order to situate it and compare it with some known ones.

- From a probabilistic point of view, since we compute the conditional law of the couple instead of the marginal ones, Theorem 3 extend both formulas (6.5.2) and (6.5.3) of Proposition 6.5.1.1 p. 373, stated for $a \geq 2$, in Jeanblanc, Yor and Chesney.[14]
- It is worth to note that, in the case $a < 2$, when μ tends to 0, we do not track down the marginal Laplace transform, since $\int_0^t \frac{ds}{X_s}$ is not finite almost surely. Indeed, in the general case $a > 0$, this marginal Laplace transform is given by formula (2.m) page 432 of Pitman and Yor,[24] namely

$$
\mathbb{E}_x \left(e^{-\lambda \int_0^t X_s \, ds} | X_t = y \right)
$$

$$
= \frac{\sqrt{2\lambda} t}{\sinh(\sqrt{2\lambda} t)} \exp\left(\frac{x + y}{2t} \left[1 - \sqrt{2\lambda} t \coth(\sqrt{2\lambda} t) \right] \right)
$$

$$
\times \frac{I_v \left(\sqrt{2\lambda xy} / \sinh(\sqrt{2\lambda} t) \right)}{I_v \left(\sqrt{xy}/t \right)} \tag{17}
$$

for all $\lambda \geq 0$, with $v = a/2 - 1$.

Now, we return to the general CIR process, starting from x, solution to (1), namely $dX_t = (a - bX_t)dt + \sqrt{2\sigma X_t} dW_t$ and we use the above result, to derive the following conditional laws.

Theorem 4.   *For a CIR process solution to (1), on the one hand, we have for* $\lambda \geq 0$

$$
\mathbb{E}_x \left(e^{-\lambda \int_0^t X_s \, ds} | X_t = y \right) = \frac{\gamma(\lambda)}{b} \frac{\sinh(bt/2)}{\sinh(\gamma(\lambda)t/2)}
$$

$$
\times \exp\left(\frac{x + y}{2\sigma} \left[b \coth(bt/2) - \gamma(\lambda) \coth(\gamma(\lambda)t/2) \right] \right)
$$

$$
\frac{I_v \left(\frac{\gamma(\lambda)}{\sigma} \sqrt{xy} / \sinh(\gamma(\lambda)t/2) \right)}{I_v \left(\frac{b}{\sigma} \sqrt{xy} / \sinh(bt/2) \right)}, \tag{18}
$$

with $\gamma(\lambda) = \sqrt{b^2 + 4\lambda\sigma}$ *and* $v = \frac{a}{\sigma} - 1$. *On the other hand, we have for* $\lambda > 0$

$$
\mathbb{E}_x \left(e^{-\lambda \int_0^t \frac{ds}{X_s}} | X_t = y \right) = \frac{I_{\gamma(\lambda)} \left(\frac{b}{\sigma} \sqrt{xy} / \sinh(bt/2) \right)}{I_v \left(\frac{b}{\sigma} \sqrt{xy} / \sinh(bt/2) \right)} \tag{19}
$$

with $\gamma(\lambda) = \sqrt{(a - \sigma)^2 + 4\lambda\sigma}/\sigma$ *and* $v = \frac{a}{\sigma} - 1$.

Proof. Let us define the process $Y_t = X_{2t/\sigma}$, by a scaling argument, we have

$$dY_t = (\alpha - \beta Y_t)dt + 2\sqrt{Y_t}dW_t, \quad Y_0 = x,$$

with $\alpha = \frac{2a}{\sigma}$ and $\beta = \frac{2b}{\sigma}$. We now write the Laplace transform

$$\mathbb{E}_x\left(\exp\left(-\lambda \int_0^t \frac{ds}{X_s}\right)|X_t = y\right) = \mathbb{E}_x\left(\exp\left(-\frac{2\lambda}{\sigma}\int_0^{\sigma t/2}\frac{ds}{Y_s}\right)|Y_{2t/\sigma} = y\right).$$

Thanks to the change of law formula (6.*d*) of Pitman and Yor,[24] we get

$$\mathbb{E}_x\left(\exp\left(-\lambda \int_0^t \frac{ds}{X_s}\right)|X_t = y\right)$$
$$= \frac{\mathbb{E}_x\left(\exp\left(-\frac{2\lambda}{\sigma}\int_0^{\sigma t/2}\frac{ds}{R_s} - \frac{b^2}{2\sigma^2}\int_0^{\sigma t/2}R_s ds\right)|R_{2t/\sigma} = y\right)}{\mathbb{E}_x\left(\exp\left(-\frac{b^2}{2\sigma^2}\int_0^{\sigma t/2}R_s ds\right)|R_{2t/\sigma} = y\right)},$$

where $(R_t)_{t \geq 0}$ is a $\mathrm{BESQ}_x^{2a/\sigma}$ process. In the same manner

$$\mathbb{E}_x\left(\exp\left(-\lambda \int_0^t X_s ds\right)|X_t = y\right)$$
$$= \frac{\mathbb{E}_x\left(\exp\left(-(\frac{2\lambda}{\sigma} + \frac{b^2}{2\sigma^2})\int_0^{\sigma t/2}R_s ds\right)|R_{2t/\sigma} = y\right)}{\mathbb{E}_x\left(\exp\left(-\frac{b^2}{2\sigma^2}\int_0^{\sigma t/2}R_s ds\right)|R_{2t/\sigma} = y\right)}.$$

We complete the proof, by combining relations (16) and (17) with the last two equations. $\square$

4.2. Numerical Results

The task now is to check the validity of the results obtained on our estimators and to understand how fast the convergence actually takes place as $T \to \infty$, with computer simulations. By relation (14), the illustrations of Theorem 1, about the MLE of b, involve the simulation of the couple $(X_T, \int_0^T X_s ds)$. Therefore, we use the exact simulation method proposed by Broadie and Kaya[3] to generate it. For Theorem 2, concerning the MLE of a, we use relation (15) and we introduce a new exact simulation method for the couple $\left(X_T, \int_0^T \frac{ds}{X_s}\right)$ based on the theoretical results of the above subsection.

4.2.1. *The MLE of* b

The method of Broadie and Kaya[3] is to simulate, at first, the random variable X_T with a noncentral chi-squared distribution

$$X_T \overset{law}{=} \frac{\sigma(1 - e^{-bt})}{2b} \chi'^2_{\frac{2a}{\sigma}} \left(\frac{2be^{-bt}}{\sigma(1 - e^{-bt})} x \right), \quad t > 0 \qquad (20)$$

where $\chi'^2_d(nc)$ denotes the noncentral chi-squared random variable with d degrees of freedom and noncentrality parameter nc. The second step is to deduce the conditional characteristic function, say Φ, of $\int_0^T X_s ds$ given X_T by setting $\lambda = -iu, u \in \mathbb{R}$, in relation (18). Then, the cumulative distribution function F of the conditional law is computed using Fourier inversion method. More precisely, we have

$$F(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\sin(ux)}{u} \Phi(u) du = \frac{2}{\pi} \int_0^{+\infty} \frac{\sin(ux)}{u} \mathrm{Re}\left[\Phi(u)\right] du.$$

This integral is approximated, using a trapezoidal rule with step discretization h, by an infinite sum that is truncated to an order N, namely

$$F(x) \simeq \frac{hx}{\pi} + \frac{2}{\pi} \sum_{j=1}^N \frac{\sin(hjx)}{j} \mathrm{Re}\left[\Phi(hj)\right].$$

The choice of parameters h and N to achieve a desired accuracy is well explained in their paper. Nevertheless, they draw attention to the continuity problem in the numerical representation of the modified Bessel function of first kind, with a complex argument, that appears in the numerator of the characteristic function Φ. In fact, the modified Bessel function of first kind characterized by the following power series

$$I_v(z) = \left(\frac{z}{2}\right)^v \sum_{j=0}^\infty \frac{\left(\frac{z^2}{4}\right)^j}{j! \Gamma(v + j + 1)},$$

where $\Gamma(x)$ is the gamma function and z is a complex number, presents a discontinuity problem in the representation of the power term z^v. Because this last function is multivalued and most of software packages consider it equal to $\exp(v \log(z))$, where $\log(z)$ is computed on the principal branch of $\arg(z)$. To avoid this difficulty, Broadie and Kaya carefully tracked $\arg(z)$ when evaluating $I_v(z)$ and changed the branch when necessary by $I_v(ze^{m\pi i}) = e^{mv\pi i} I_v(z)$, where m is an integer value. Recently, Lord and Kahl[21] showed how to avoid this complex discontinuity problem. They considered, up to a scaling coefficient, the complex-valued argument in the modified Bessel function, $z(u) = \gamma(u)/\sinh(\gamma(u)t)$,

where $\gamma(u) = \sqrt{b^2 - 4\sigma iu}$ and evaluated the characteristic function as $\Phi(u)e^{v \log(z(u))}/z(u)^v$, where $\Phi(u)$ and $z(u)^v$ are evaluated using the principal branch, however, $\log(z(u))$ is evaluated as $-\gamma(u)t/2 + \log(\gamma(u)/(1 - e^{\gamma(u)t}))$, the last logarithm term is evaluated on its principal branch. This correction term introduced by Lord and Kahl[21] improves considerably our simulations.

The graphical representations of Figure 1 illustrate the limit law of the MLE $\hat{b}_T$ of b, stated in Theorem 1. In order to do that, we simulate N independent trajectories of the normalized error, with different values of parameters (x, a, b, c, T) to cover the various cases, and we plot their histogram. When the limit is Gaussian, we normalize the error by the appropriate term to compare the histogram with the standard Gaussian density. Otherwise, since we do not have an explicit formula of the limit law density, we simply plot the histogram of the error as stated in the theorem. Both histograms, on the top of Figure 1, deal with ergodic and nonergodic cases, for $b > 0$. Those at the bottom, treat the cases $b = 0$ and $b < 0$. Note that in the latter cases, we choose $a < \sigma$ to cover the subtle case when the CIR can reaches the state 0.
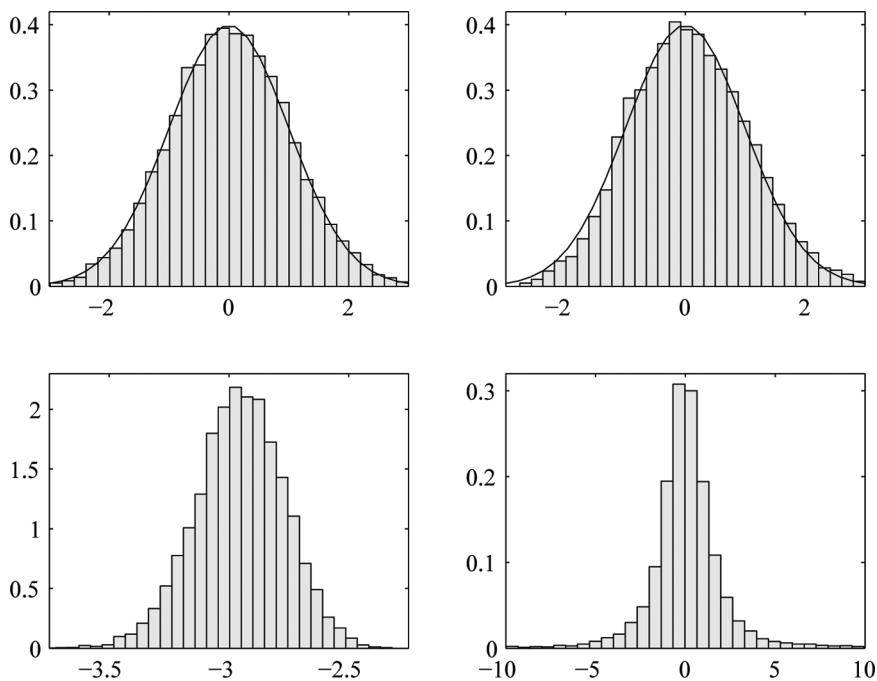

FIGURE 1 Convergence in distribution of the error, $\hat{b}_T - b$, correctly normalized, in different cases, for a sample $N = 10000$, from the left to the right, we have (x, a, b, σ, T) equal to $(1, 2, 1, 1, 1e+3)$, $(1, 0.5, 1, 1, 1e+3)$, $(1, 0.75, 0, 1, 1e+2)$ and $(1, 0.75, -1, 1, 12)$.

Remark.   From a practical point of view, note that when the theoretical rate of convergence is of order T (resp. T^2) the limit distribution is well approximated from $T = 1e + 3$ (resp. $T = 1e + 2$). However, when the theoretical rate of convergence is exponential then a stabilization is observed only from $T = 12$ (see Figure 1).

4.2.2.   *The MLE of* a

We proceed to simulate the couple $\left(X_T, \int_0^T \frac{ds}{X_s}\right)$ in the same manner as below. First, we begin by generating a random variable X_T using the property (20). Then, in order to simulate $\int_0^T \frac{ds}{X_s}$, we use relation (19) to compute the cumulative distribution function of the conditional law. To do that, we use the Fourier inversion method introduced previously.

The graphical representation below illustrates the limit law of the MLE $\hat{a}_T$ of a stated in Theorem 2. We simulate N independent trajectories of the correctly normalized error, as explained in the subsection above, and we plot their histogram. Both histograms, on the top of Figure 2, deal with ergodic and nonergodic cases, for $a > \sigma$. Those at the bottom treat cases $b > 0$ and $b = 0$, for $a = \sigma$.
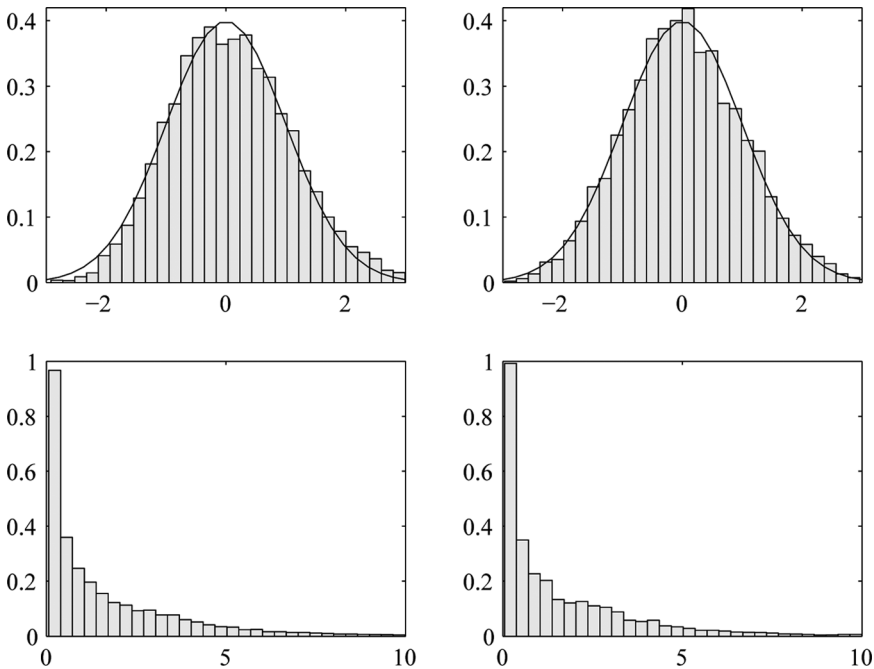

FIGURE 2   Convergence in distribution of the error, $\hat{a}_T - a$, correctly normalized, in different cases, for a sample $N = 10000$, from the left to the right, we have (x, a, b, σ, T) equal to $(1, 2, 0, 1, 1e + 100)$, $(1, 2, 1, 1, 1e + 3)$, $(1, 1, 0, 1, 1e + 50)$ and $(1, 1, 1, 1, 1e + 2)$.

Remark.   In this case, note that when the theoretical rate of convergence is of order $\sqrt{T}$ (resp. T) the limit distribution is well approximated from $T = 1e + 3$ (resp. $T = 1e + 2$). However, when the theoretical rate of convergence is of order $\sqrt{\log T}$ (resp. $\log T$) then to observe a stabilization $T = 1e + 100$ (resp. $T = 1e + 50$) was needed (see Figure 2).

5. CONCLUSION

The precise description of the behavior of $\int_0^T X_s ds$ and $\int_0^T \frac{ds}{X_s}$ established in the present article provides a new approach to overcome the problem of parameters estimation for the CIR model in all its generality. When we estimate one of the drift parameter and suppose known the other one, we obtain original results that are confirmed by exact simulation methods. A natural question is now the problem of the global estimation for the CIR model. Answering this question involves more complicated calculations and this is the object of a forthcoming work.

ACKNOWLEDGMENTS

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